ISSN 2228-9860 eISSN 1906-9642 CODEN: ITJEA8

International Transaction Journal of Engineering, Management, & Applied Sciences & Technologies

http://TuEngr.com

The Solution of the First Main Problem of the Theory of Elasticity for a Transtropic Body of Revolution

Dmitry A. Ivanychev[1*], Ekaterina Yu. Levina[2], Evgeniy A. Novikov[1], Maxim V. Polikarpov[1]

[1] *Department of General Mechanics, Lipetsk State Technical University, Lipetsk, RUSSIA.*

[2] *Department of Physics, Bauman Moscow State Technical University, Moscow, RUSSIA.*

**Corresponding Author (Email: Lsivdmal@mail.ru).*

Paper ID: 12A3L

Volume 12 Issue 3

Received 01 October 2020

Received in revised form 14 December 2020

Accepted 05 January 2021

Available online 15 January 2021

Keywords:

Boundary state method; Transverse isotropy; Spatial problems; Axisymmetric problems; Isomorphism of state spaces; Elasticity for anisotropic bodies.

Abstract

Based on the method of boundary states, this work investigates the elastic equilibrium of transversely isotropic bodies of revolution under the action of forces applied to the surface of a body. A new method is proposed for the formation of bases of internal and boundary states based on a general solution of plane deformation and formulas for the transition to a spatial axisymmetric state. Isomorphism of state spaces is proved. Isomorphism of spaces allows the search for an internal state to be reduced to the study of boundary states. In the case of the first and second main problems of the theory of elasticity, the problem is reduced to expanding the sought state in a series in terms of the orthonormal elements and finding the Fourier coefficients of this linear combination. Fourier coefficients are quadratures. The first main problem of the theory of elasticity for a transversely isotropic hemisphere with boundary conditions imitating inhomogeneous tension is solved. Verification of the solution is presented and accuracy is assessed. The results are presented graphically.

Disciplinary: Engineering Mechanics.

Cite This Article:

Ivanychev, D. A., Levina, E.Yu., Novikov, E.A., Polikarpov, M.V. (2021). The Solution of the First Main Problem of the Theory of Elasticity for a Transtropic Body of Revolution. *International Transaction Journal of Engineering, Management, & Applied Sciences & Technologies, 12*(3), 12A3L, 1-9. http://TUENGR.COM/V12/12A3L.pdf DOI: 10.14456/ITJEMAST.2021.54

1 Introduction

In modern mechanical engineering, materials that are interpreted by the theory of elasticity as anisotropic with respect to elastic properties are increasingly used. These include polymers, fiberglass reinforced plastics, polycrystalline metals, rocks, etc. This fact requires the creation of new or development of existing methods for calculating the strength of solids made of such materials.

In the theory of elasticity, problems for anisotropic bodies were considered less often than problems for isotropic bodies, but at present, a lot of works have been solved for solving boundary value problems of the theory of elasticity for anisotropic bodies with different degrees of symmetry. For example, for a transversely isotropic cylinder, the first main problem of the theory of elasticity [1], the second main problem [2], and the contact problem without friction in the contact area [3] with the participation of mass forces are solved. The peculiarity of the problems is that the elastic field simultaneously satisfies the conditions on the boundary and inside the region (mass forces).

In [4], boundary value problems of thermoelasticity were solved for transtropic bodies of revolution, where the resulting field is the sum of three elastic fields: from the boundary value problem of elastostatics, in the problem from the action of only mass forces, and in the thermoelastic problem.

In [5], using the general representation of S. G. Lehnitsky, the problem of torsion of an extended anisotropic cylinder was solved. The solution was carried out using the boundary state method.

The study of the deformation of anisotropic bodies has always been an actual direction in science. External forces can simulate not only uniaxial tension but also torsion of solid and hollow anisotropic cylinders. In the working [5], a variant of flow theory was developed for the case of materials with large anisotropic elastoplastic deformations. The corresponding dynamic problem was formulated and a numerical method for solving two-dimensional axisymmetric problems was developed. Many papers devoted to the torsion of bodies from anisotropic materials, for example, in the working [6], the stress-strain state of anisotropic cylindrical and prismatic rods was investigated under an arbitrary plasticity condition. In [7], a study of the stresses distribution peculiarities was undertaken and displacements in individual layers of a multilayer anisotropic rod. In the working [8], a method is proposed for solving the problem of layered anisotropic rod torsion by the finite element method. The problem of rods torsion of rhomboid section and section of the compressor is considered. In the working [9], solutions analysis for torsion problems and stretching of nanotubes with two types of cylindrical anisotropy is given, which was theorized by S.G. Lehnitsky in the framework of the classical theory of elasticity.

The present work is devoted to the development of the method of boundary states for the class of solutions of basic problems of the theory of elasticity for transversely isotropic bodies of revolution.

2 Formulation of the Problem

The elastic equilibrium of a transversely isotropic body bounded by one or several coaxial surfaces of revolution is considered. At the boundary of the body (Figure 1), forces p_{r0}, p_{z0} (first main task) or displacements u_0, w_0 (second main task) can be specified. There are no mass forces. All external conditions are axisymmetric about the axis of rotation z.

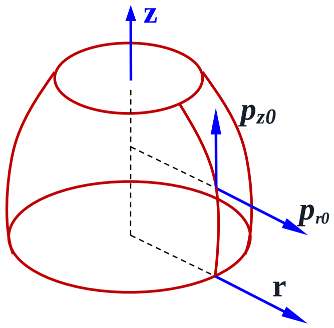

Figure 1: The transversely isotropic body of revolution.

The task is to find the stress-strain state that occurs in the body under the influence of given factors.

3 The Solution Method

To solve this problem, we use the boundary state method (BSM) [11]. BSM is a new energy method for solving problems of equations of mathematical physics. The method demonstrated its efficiency in solving boundary problems of the elasticity theory, both for isotropic and anisotropic media, in solving problems of thermoelasticity, hydrodynamics of an ideal fluid, dynamics (oscillations) of isotropic bodies.

The foundation of the method is the space of internal Ξ and boundary Γ states:

$$\Xi = \{\xi_1, \xi_2, \xi_3, ..., \xi_k, ...\}; \quad \Gamma = \{\gamma_1, \gamma_2, \gamma_3, ..., \gamma_k, ...\}.$$

The internal state is determined by the sets of displacements vector components, tenors of deformations, and stresses:

$$\xi_k = \{u_i^k, \varepsilon_{ij}^k, \sigma_{ij}^k\}. \tag{1}$$

The main difficulty in forming a solution in the BSM is the design of the basis for internal states, that relies on a common or fundamental solution for the environment; It is also possible to use any private or special solutions. The method of constructing the basis of internal states will be described below.

Scalar product in the space of internal states Ξ is expressed through the internal energy of elastic deformation (hence the membership of the method in the energy class). For example, for the first and second internal state of the body occupying the V region:

$$(\xi_1, \xi_2) = \int_V \varepsilon_{ij}^1 \sigma_{ij}^2 \, dV,$$

moreover, due to the commutatively of the states of the medium:

$$(\xi_1, \xi_2) = (\xi_2, \xi_1) = \int_V \varepsilon_{ij}^1 \sigma_{ij}^2 \, dV = \int_V \varepsilon_{ij}^2 \sigma_{ij}^1 \, dV.$$

The boundary state is determined by the components of the vector of displacement of the points of the boundary and surface forces:

$$g_k = \{u_i^k, p_i^k\}; \quad p_i^k = \sigma_{ij}^k n_j,$$

where n_j – is a component of the normal to the boundary.

In the space of boundary states *Γ*, the scalar product expresses the work of external forces on the surface of the body *S*, for example, for the first and second states:

$$(g_1, g_2) = \int_S p_i^1 u_i^2 \, dS \; .$$

moreover, under the principle of possible movements:

$$(g_1, g_2) = (g_2, g_1) = \int_S p_i^1 u_i^2 \, dS = \int_S p_i^2 u_i^1 \, dS \; .$$

It is proved that in the case of a smooth boundary both state spaces are Hilbert and are conjugated by an isomorphism [11]. By definition, each element $x_k \in X$ corresponds to a single element $g_k \in \Gamma$, and this relationship exists on a one-to-one basis: $x_k \leftrightarrow g_k$. This allows the internal state search to be reduced to the construction of a boundary state that is isomorphic to it. The latter essentially depends on the boundary conditions. In the case of the first and second main problems of mechanics, the problem recedes to resolving a system of equations for the Fourier coefficients, decomposition of the desired inner x and boundary g states in a series in terms of the orthonormal basis elements:

$$x = \sum_{k=1}^{\infty} c_k x_k \; ; \quad g = \sum_{k=1}^{\infty} c_k g_k \; .$$

or explicitly:

$$p_i = \sum_{k=1}^{\infty} c_k p_i^k \; ; \quad u_i = \sum_{k=1}^{\infty} c_k u_i^k \; ; \quad \sigma_{ij} = \sum_{k=1}^{\infty} c_k \sigma_{ij}^k \; ; \quad \varepsilon_{ij} = \sum_{k=1}^{\infty} c_k \varepsilon_{ij}^k \; .$$

The Fourier coefficients in the case of the first main problem of the theory of elasticity, when forces $\mathbf{p} = \{p_{r0}, p_{z0}\}$ are set on the body surface, are calculated as follows:

$$c_k = (\mathbf{p}, \mathbf{u}^k) = \int_S (p_{r0} u^k + p_{z0} v^k) \, dS$$

where p_{r0}, p_{z0} – set at the end of the effort, $\mathbf{u}^k = \{u^k, v^k, w^k\}$ – displacement vector in the basic element $g_k = \{u_i^k, p_i^k\}$.

In the case of the second main task, when displacements $\mathbf{u} = \{u_0, w_0\}$ are specified on the surface of the body, they are calculated as follows:

$$c_k = (\mathbf{u}, \mathbf{p}^k) = \int_S (u_0 p_r^k + w_0 p_z^k) \, dS$$

where u_0, w_0 – given components of the displacement vector at the boundary, $\mathbf{p}^k = \{p_r^k, p_z^k\}$ is the force vector in the basis of boundary states $g_k = \{u_i^k, p_i^k\}$.

4 Construction of the Basis of Internal States

In [12], based on the method of integral superposition, a relationship was established between the spatial stressed and deformed states of an elastic transversely isotropic body and certain auxiliary two-dimensional states, the components of which depend on two coordinates z and y (variables). Plane deformation arising in cylinders with an elastic symmetry plane at each point parallel to the zy plane (direction η) is used as plane auxiliary states [2]:

$$
\begin{aligned}
&\sigma_z^{pl} = -\operatorname{Re}[\gamma_1^2\varphi_1'(V_1)+\gamma_2^2\varphi_2'(V_2)];\\
&\sigma_y^{pl} = \operatorname{Re}[\varphi_1'(V_1)+\varphi_2'(V_2)];\\
&\sigma_{zy}^{pl} = -\operatorname{Re}[\gamma_1\varphi_1'(V_1)+\gamma_2\varphi_2'(V_2)];\\
&\sigma_\eta^{pl} = \nu_r\sigma_y^{pl}+\nu_z\frac{E_r}{E_z}\sigma_z^{pl};\quad \tau_{z\theta}=0;\quad \tau_{r\theta}=0;\\
&u_z^{pl} = \operatorname{Re}[p_1\varphi_1(V_1)+p_2\varphi_2(V_2)];\\
&u_y^{pl} = \operatorname{Re}[iq_1\varphi_1(V_1)+iq_2\varphi_2(V_2)],
\end{aligned}
\tag{2}
$$

where the constants q_1 and p_1 are determined by the elastic parameters of the material, $V_j = z/\gamma_j + iy$, γ_j are the complex roots of the characteristic equation:

$$\left(1-\nu_z^2\frac{E_r}{E_z}\right)\gamma_j^4-\left[\frac{E_z}{G_z}-2\nu_z(1+\nu_r)\right]\gamma_j^2+(1-\nu_r^2)\frac{E_z}{G_r}=0,$$

functions $\varphi_j(V_j)$ – analytic in their variables. Using relation (2), the transition to an axisymmetric spatial state in cylindrical coordinates is carried out [12]:

$$
\begin{aligned}
&\sigma_z=\frac{1}{\pi}\int_{-r}^{r}\frac{\sigma_z^{pl}}{r\sqrt{r^2-y^2}}dy;\quad \sigma_{zr}=\frac{1}{\pi}\int_{-r}^{r}\frac{\sigma_{zy}^{pl}}{r\sqrt{r^2-y^2}}dy;\quad \sigma_{z\theta}=\sigma_{r\theta};\\
&\sigma_r-\sigma_\theta=\frac{1}{\pi}\int_{-r}^{r}\frac{(\sigma_y^{pl}-\sigma_\eta^{pl})(2y^2-r^2)}{r^2\sqrt{r^2-y^2}}dy;\quad \sigma_r+\sigma_\theta=\frac{1}{\pi}\int_{-r}^{r}\frac{(\sigma_y^{pl}+\sigma_\eta^{pl})}{\sqrt{r^2-y^2}}dy;\\
&u=\frac{1}{\pi}\int_{-r}^{r}\frac{u_y^{pl}}{r\sqrt{r^2-y^2}}dy;\quad w=\frac{1}{\pi}\int_{-r}^{r}\frac{u_z^{pl}}{r\sqrt{r^2-y^2}}dy;\quad v=0.
\end{aligned}
\tag{3}
$$

The basis sets (1) can be constructed by generating possible variants for the two analytic functions $\varphi_1(V_1)$ and $\varphi_2(V_2)$ of the flat auxiliary state (2). General view of the representation of analytical functions for a simply connected bounded domain:

$$\varphi_j(V_j)=\sum_{n=0}^{\infty} a_{jn}V_j^n, \quad (j=1,2).$$

In this case, the basis of the spaces of internal states is made up of the sets:

$$\begin{pmatrix}\varphi_1(V_1)\\ \varphi_2(V_2)\end{pmatrix} \in \left\{ \begin{pmatrix}V_1^n\\ 0\end{pmatrix}, \begin{pmatrix}0\\ V_2^n\end{pmatrix}, \begin{pmatrix}iV_1^n\\ 0\end{pmatrix}, \begin{pmatrix}0\\ iV_2^n\end{pmatrix}, \ldots \right\}, n=1,2,\ldots \quad (4)$$

According to (4), all the elastic characteristics of the plane auxiliary state are determined, and then a transition to a three-dimensional state follows according to dependences (3).

Recent studies (e.g. [13]) make it possible to extend the class of regular solutions to the class of solutions containing mechanical singularities. The centers of expansion, concentrated forces applied to the boundary of the body can serve as such. The approach of Alexandrov and Solov'ev [12] allows one to include singular solutions with singularities along with the circles.

5 Result Analysis

Let us consider the equilibrium of a transversely isotropic body in the form of a hemisphere (Figure 2) from a large dark gray siltstone rock [14, 15]. After the dimensioning procedure, the elastic characteristics of the material: $E_z=6.21$; $E_r=5.68$; $G_z=2.55$; $\nu_z=0.22$; $\nu_r=0.24$ and body region $D=\{(z,r)|\ 0\le r\le 1,\ 0\le z\le 1\}$.

The load S_1 is applied to the border of the $p_1=-2/3$ hemisphere; a tensile load along the z-axis is applied to the boundary S_2: $p_2=1-r^2$ (Figure 2).

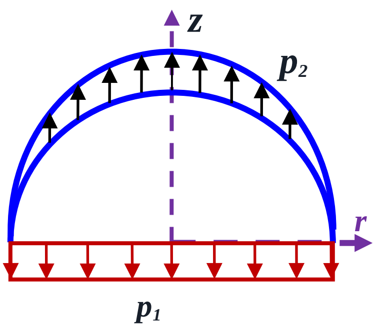

Figure 2: Boundary conditions to the problem for the hemisphere

After constructing the basis of internal states (1), it is orthogonalized according to the recursive matrix algorithm [16].

To restore the elastic field, a basis of 30 elements was required. Checking the result and assessing the accuracy of the solution is carried out by comparing the specified boundary conditions with those reconstructed as a result of the solution (Figure 3). Hereinafter, the given ( ) and reconstructed ( ) boundary conditions are shown to scale on the graphs. For example, the true value p_r in the upper left graph of Figure 3 is equal to the value on the graph multiplied by a factor of k.

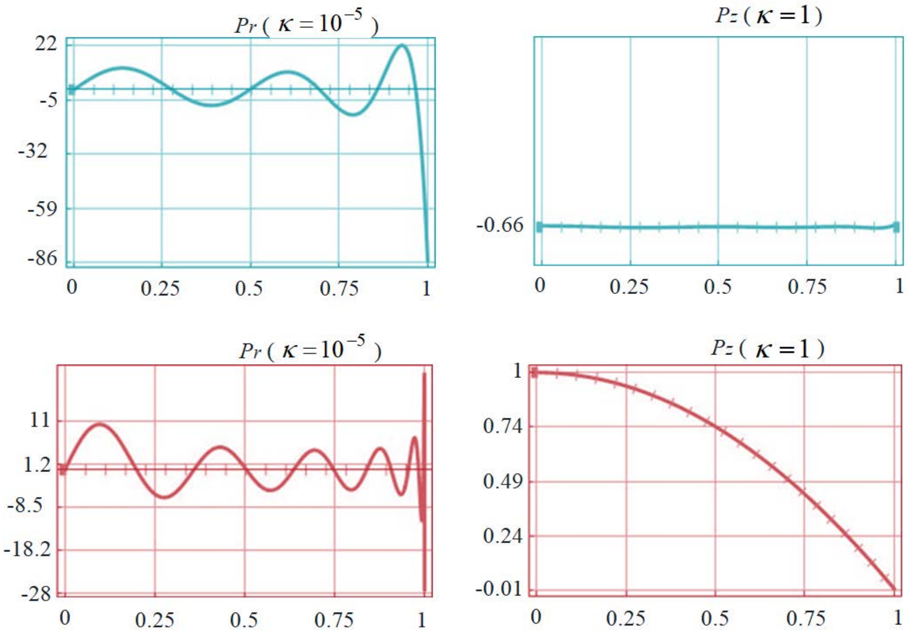

Figure 3: Verification of boundary conditions at sections S_1 and S_2

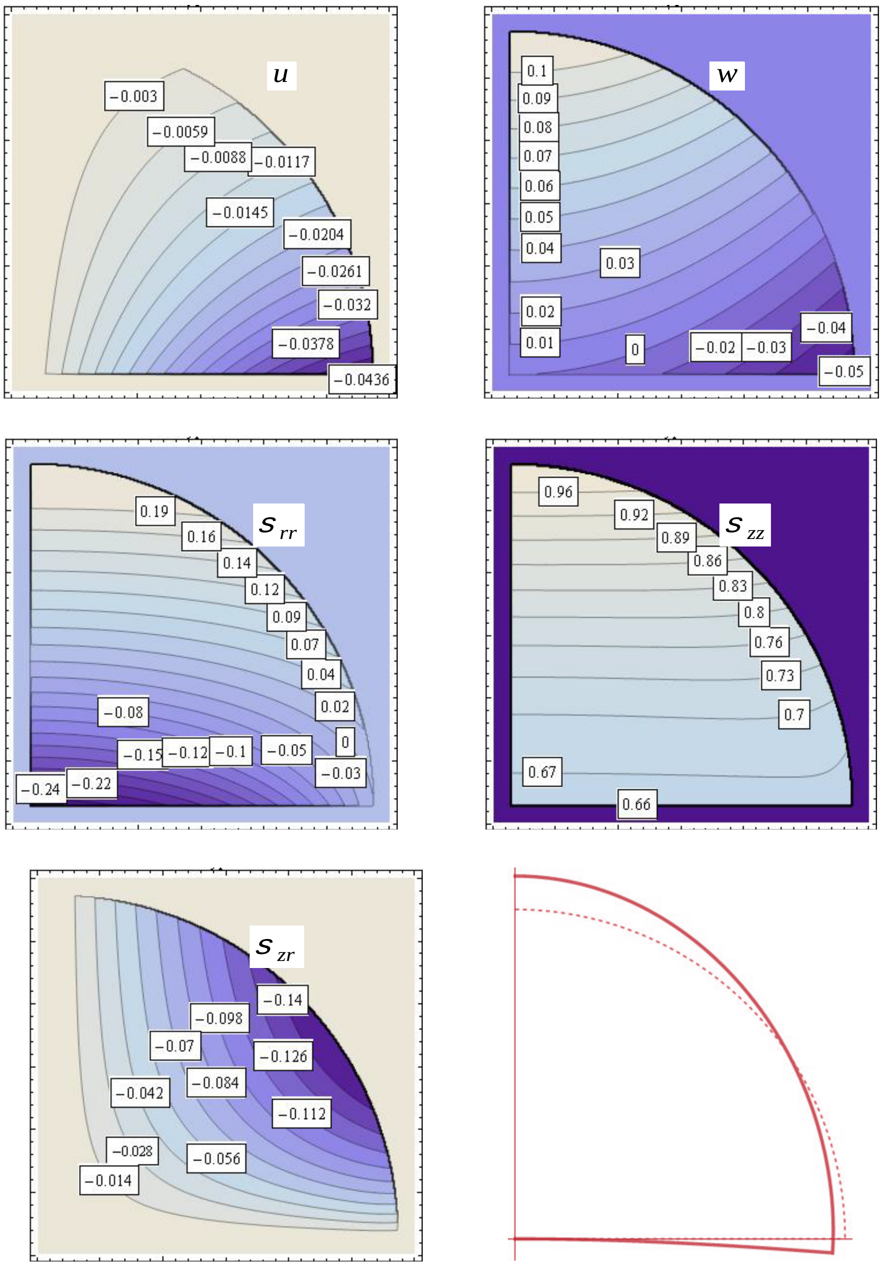

Figure 4: Isolines of the elastic field components and the contour of the deformed body.

From the graphs Figure 3, the efforts at the border of the body are restored with high accuracy. This indicates good convergence of the solution. The maximum error was 1% (for effort p_z on the lower right graph at points $r=1,\ z=0$).

The isolines of the obtained components of the elastic field are shown in Figure 4 (the expressions are cumbersome and explicitly immense). Due to axial symmetry, the area $\{(z,r)|\ 0\le r\le 1,\ 0\le z\le 1\}$ is shown. For all isolines $k=1$. The contour of the deformed state (Figure 4) is shown in an exaggerated form (dash-dotted line – the original contour, solid – deformed).

The advantage of the method is that the orthonormal basis of the spaces of internal and boundary states is constructed once for a body made of a certain material and can be used to solve problems with different boundary conditions.

6 Conclusion

Thus, the method of boundary states has shown its effectiveness in determining the axisymmetric stress-strain state of transversely isotropic bodies of revolution under the conditions of the first and second main problems of the theory of elasticity. The convergence of the solution is ensured by increasing the basis of internal states. Exceptions are special border points.

7 Availability of Data and Material

Data can be made available by contacting the corresponding author.

8 Acknowledgment

The reported study was funded by RFBR under the project No.19-31-90065 and the Lipetsk Region under the research project No.19-41-480003"r_a".

9 References

[1] Ivanychev, D. A. (2020). The method of boundary states in the solution of the first fundamental problem of the theory of anisotropic elasticity with mass forces. *Tomsk State University Journal of Mathematics and Mechanics*. No. 66. pp.96-111. DOI 10.17223/19988621/66/8.

[2] Ivanychev, D. A. (2019). The method of boundary states in the solution of the second fundamental problem of the theory of anisotropic elasticity with mass forces. *Tomsk State University Journal of Mathematics and Mechanics*, No. 61, pp. 45-60. DOI 10.17223/19988621/61/5.

[3] Ivanychev, D. A. (2019). The contact problem Solution of the elasticity theory for anisotropic rotation bodies with mass forces. *PNRPU Mechanics Bulletin*, No. 2, pp. 49-62. DOI: 10.15593/perm.mech/2019.2.05.

[4] Ivanychev D. A., Levina E. Yu. (2019). Solution of thermo elasticity problems for solids of revolution with transversal isotropic feature and a body force. *Journal of Physics: Conference Series, 1348*(012058), 15p. DOI: 10.1088/17426596/1348/1/012058.

[5] Ivanychev, D. A., Levina, E. Yu., Abdullakh, L. S., Glazkova, Yu. A. (2019) The method of boundary states in problems of torsion of anisotropic cylinders of finite length. *International Transaction Journal of Engineering, Management, & Applied Sciences & Technologies, 10*(2), 183-191. DOI: 10.14456/ITJEMAST.2019.18.

[6] Dimitrienko, Yu. I. (2003). The anisotropic theory of finite elaatic-plastic deformations. *Vestnik MGTim. N.E. Bauman. Ser., "Natural Sciences"*, No 2. pp. 47-59.

[7] Mironov B. G., Derevyannykh E. A. (2012). On the general relations of the theory of torsion of anisotropic rods. *Vestnik ChGPU im. I. Ya. Yakovlev, 4*(76). pp. 108-112.

[8] Nurimbetov A. U. (2009). Torsion of a multi-layered prismatic anisotropic rod composed of orthotropic materials. Vestnik RUDN Series Math. Computer science. *Physics*, No.4, pp.63-75.

[9] Nurimbetov A. U. (2015). Stress-strain state of layered composite rods and blades during torsion. *Construction mechanics of engineering structures and structures*, No. 1, pp.59-66.

[10] Goldstein R. V., Gorodtsov V. A., Lisovenko D. S. (2009). To the description of multilayer nanotubes within the framework of cylindrical anisotropic elasticity models. *Physical mesomechanics*, *12*(5), 5-14.

[11] Penkov V. B. (2001). The method of boundary states for solving problems of linear mechanics. *Far eastern mathematical journal*, *2*(2), 115-137.

[12] Aleksandrov A. Y., Solovyev Y. I. (1978). *Prostranstvennye zadachi teorii uprugosti (primenenie metodov teorii funkcij kompleksnogo peremennogo*. Nauka, 464p.

[13] Polikarpov, M. V., Penkov, V. B. (2020). Method of boundary states in problems of interactions of two cavities. *International Transaction Journal of Engineering, Management, & Applied Sciences & Technologies*, *11*(12), 220-229. DOI: 10.14456/ITJEMAST.2020.244.

[14] Lekhnitskiy, S. G. (1957). *Anisotropic plates*. GITTL, 463p.

[15] Lekhnitskiy S. G. (1977). *Theory of elasticity of the anisotropic body*. Nauka, 416p.

[16] Satalkina L. V. (2007). Increasing the basis of the state space under strict restrictions to the energy intensity of calculations. Lipetsk: LSTU. *Collection of abstracts of the scientific conference of students and graduate students of Lipetsk State Technical University*, pp.130–131.

Dr.Dmitry Alekseevich Ivanychev is an assistant professor at Lipetsk State Technical University. He holds a PhD in Engineering Mechanics. His works are in the areas of Theoretical Mechanics and Strength of Materials.

Dr. Ekaterina Yur'yevna Levina is an Assistant Professor at the Department of Physics, Bauman Moscow State Technical University. She holds a PhD. Her research encompasses Mechanics.

Evgeniy Aleksandrovich Novikov is a postgraduate student at Lipetsk State Technical University. His research encompasses Solid Mechanics.

Maxim Vladimirovich Polikarpov is a graduate student at the Department of General Mechanics, Lipetsk State Technical University, Lipetsk, Russian Federation. His research encompasses Solid Mechanics.