

INNOVATIVE EXPLICIT FINITE DIFFERENCE SOLUTION TO TIME RATE OF SETTLEMENT IN CLAY

AMIR AL-KHAFAJI

Dept of Civil Engineering & Construction, Bradley University, Peoria, USA

The time rate of settlement process in clay involves changes in excess pore water pressure (EPWP) with time. Exact solutions have been published for constant initial EPWP and other simple distributions. The finite differences methods are generally used in solving complex initial EPWP distributions. Such methods suffer from roundoff errors at each time increment and truncation errors proportional to the step size used. The explicit finite difference method produces stable solutions when proper time and depth increments are used. An innovative explicit finite difference model involving eigenvalues and eigenvectors is proposed that will permit arbitrary initial EPWP distributions and reduce roundoff errors. This method is numerically stable and convergent. Unlike traditional methods, the proposed solution will also eliminate the need to calculate the EPWP vector traditionally required at each time increment. Instead, the EPWP can be computed directly for any number of time increments.

Keywords: Exact solution, Excess water pressure, Clay soil, Roundoff errors, Truncation errors, Implicit finite difference, Eigenvalues, Step size.

1 BACKGROUND

Although analytical methods have and will continue to provide useful solution, they cannot yield realistic answers for problems involving nonhomogeneous and/or anisotropic materials with arbitrary boundary and/or initial conditions. While a complete treatment of this broad subject is beyond the scope of this paper, the one-dimensional time rate of settlement problem with arbitrary initial excess porewater pressure (EPWP) is discussed. The one-dimensional model, which relates EPWP distributions to depth at any time (Terzaghi 1943) is given as

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} \quad (1)$$

where u is the excess pore water pressure at time t and depth z and c_v is the coefficient of consolidation. Terzaghi and and Frolich (1936) presented a closed-form solution to this problem that included several simplifying assumptions was applied to a few initial EPWP distributions with depth. For most practical problems, the initial EPWP distribution is non-linear with depth due to partial EPWP dissipation under previous load and/or complex surface load configurations. Consequently, numerical methods are generally used for time rate of settlement calculations.

2 THE FINITE DIFFERENCE METHOD

The basic concept involves discretization of arbitrary continuous functions and replacing them with their equivalent difference expressions. The solution of Eq. (1) at a given time $t = t_j$ and depth z_i with an arbitrary function $u(z_i, t_j)$ at node z_i can be approximated using difference expressions available in literature (Al-Khafaji and Tooley 1986) as follows:

$$\left. \frac{\partial^2 u}{\partial z^2} \right|_{at\ t_j, z_i} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta z)^2} \quad (2a)$$

The approximation of the first derivative of u with respect to time at $t = t_j$ and $z = z_i$ is made using a forward difference approximation as:

$$\left. \frac{\partial u}{\partial z} \right|_{at\ t_j, z_i} = \frac{-u_{i,j} + u_{i,j+1}}{\Delta t} \quad (2b)$$

Substituting Eq. (2a) and Eq. (2b) into Eq. (1) yields:

$$\frac{-u_{i,j} + u_{i,j+1}}{\Delta t} = c_v \frac{(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})}{(\Delta z)^2} \quad (3)$$

For a given soil layer of thickness H_0 , the depth increment $\Delta z = H_0/m$ and the time increment $\Delta t = t/n$. Where m is the number of depth increments and n is the number of time increments for which EPWP is computed. In general, $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$. Denoting $\alpha = c_v \Delta t / \Delta z^2$, then simplifying and rearranging Eq. (3) yields:

$$u_{i,j+1} = \alpha u_{i-1,j} - (1 - 2\alpha)u_{i,j} + \alpha u_{i+1,j} \quad (4)$$

Eq. (4) is an explicit finite difference recurrence formula, which permits direct step-by-step evaluation of the EPWP. The implication is that knowing the initial and boundary EPWP values at $t = 0$, it is possible to calculate the EPWP for a given time increment $t = \Delta t$. Using Eq. (3), a set of linear algebraic equations can be developed which can be expressed in a matrix form for the $m+1$ nodes along the depth axis z as follows:

$$\begin{Bmatrix} u_{top} \\ u_1 \\ \vdots \\ u_{m-1} \\ u_{bot} \end{Bmatrix}_{t=t_{j+1}} = \begin{bmatrix} \alpha & 1-2\alpha & \alpha & & & \\ & \alpha & 1-2\alpha & \alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & 1-2\alpha & \alpha \\ & & & & \alpha & 1-2\alpha & \alpha \end{bmatrix} \begin{Bmatrix} u_{top} \\ u_1 \\ \vdots \\ u_{m-1} \\ u_{bot} \end{Bmatrix}_{t=t_j} \quad (5a)$$

Note that the EPWP drops to zero for doubly drained layer after the first time increment. That is, at $t = t_1$ we have $u_{top} = u_{bot} = 0$ and Eq. (5a) is simplified to the following:

$$\begin{Bmatrix} u_1 \\ \vdots \\ u_{m-1} \end{Bmatrix}_{t=t_{j+1}} = \begin{bmatrix} 1-2\alpha & \alpha & & & \\ \alpha & 1-2\alpha & \alpha & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha & 1-2\alpha & \alpha \\ & & & \alpha & 1-2\alpha \end{bmatrix} \begin{Bmatrix} u_1 \\ \vdots \\ u_{m-1} \end{Bmatrix}_{t=t_j} \quad (5b)$$

Eq. (5b) can be written more conveniently in a compact matrix form as:

$$\{u\}_{j+1} = [A]\{u\}_j \quad (6)$$

Eq. (6) is the explicit finite difference solution to the time rate of settlement problem. The range of α should be between 0.0 and 0.5 for the solution to be stable, in fact $\alpha = 1/6$ gives the most accurate results (Scott 1963). It is evident that the limitation imposed on α makes it necessary that the EPWP be evaluated at extremely large number of time increments. This is precisely why the finite difference method is time consuming and subject to roundoff errors.

3 THE EIGENPROBLEM METHOD

Suppose that the EPWP initially at $t = 0$ is given by the vector $\{u\}_0$, then the new values of EPWP at times $j = 0, \dots, n$ are computed explicitly using Eq. (6):

$$\{u\}_1 = [A]\{u\}_0 \quad (7)$$

$$\{u\}_2 = [A]\{u\}_1 = [A][A]\{u\}_0 = [A]^2\{u\}_0 \quad (8)$$

$$\{u\}_3 = [A]\{u\}_2 = [A][A]^2\{u\}_0 = [A]^3\{u\}_0 \quad (9)$$

and for the n^{th} increment:

$$\{u\}_n = [A]^{n-j}\{u\}_j \text{ for } j = 0, \dots, n \quad (10)$$

Eq. (10) has the advantage over Eq. (6) in that roundoff errors and error propagation at each time step are minimized. Instead, matrix $[A]$ is raised to the desired power directly without the need for computing intermediate EPWP values. The eigenvalues and their corresponding vectors for the $[A]$ matrix are determined by solving the following equation:

$$[A]\{\varphi\} = \lambda\{\varphi\} \quad (11)$$

Where λ represents the eigenvalues of the square matrix $[A]$ whose size is $(m-1) \times (m-1)$. This is because there are m nodes with depth and the EPWP at the top and bottom are eliminated due to free drainage. Hence, the solution for the $\{\varphi\}$ vector is achieved by forcing the following determinant to zero to yield the $m-1$ eigenvalues. Thus:

$$|[A] - \lambda[I]| = 0 \quad (12)$$

Expanding the determinant given by Eq. (12) yields a polynomial of order $(m-1)$ and whose roots are the eigenvalues. The evaluation of the eigenvalues is covered in most textbooks on numerical method. Fortunately, for this special case, it can be shown that the eigenvalues of matrix $[A]$ appearing in Eq. (11) are given directly by:

$$\lambda_r = 1 - 4\alpha \left[\sin^2 \left(\frac{r\pi}{2m} \right) \right] \quad \text{for } r = 1, \dots, m-1 \quad (13)$$

Where r is the interior node for which the eigenvalue is needed. Eq. (13) is to be used only when the consolidating layer is free draining at both boundaries. Substituting each of the eigenvalues into Eq. (12) yields the eigenvectors $\{\varphi\}_1, \{\varphi\}_2, \dots, \{\varphi\}_{m-1}$. Since free drainage is assumed at the top and bottom of the soil layer, there are $m-1$ depths (nodes) at which the EPWP needs to be computed. Thus:

$$\begin{aligned} [A]\{\varphi\}_1 &= \lambda_1\{\varphi\}_1 \\ [A]\{\varphi\}_2 &= \lambda_2\{\varphi\}_2 \\ &\vdots \\ [A]\{\varphi\}_{m-1} &= \lambda_{m-1}\{\varphi\}_{m-1} \end{aligned} \quad (14)$$

These equations can be expressed more conveniently in the following form:

$$[A][\{\varphi\}_1\{\varphi\}_2 \dots \{\varphi\}_{m-1}] = [\{\varphi\}_1\{\varphi\}_2 \dots \{\varphi\}_{m-1}] \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_{m-1} \end{bmatrix} \quad (15)$$

in a compact matrix form, we have:

$$[A][\phi] = [\phi][\lambda] \quad (16)$$

where $[\phi]$ is a square eigenvectors matrix and $[\lambda]$ is a diagonal eigenvalues matrix. Therefore, multiplying Eq. (16) by the inverse of the eigenvector matrix $[\phi]^{-1}$ gives:

$$[A] = [\phi][\lambda][\phi]^{-1} \quad (17)$$

The square of the matrix $[A]$ is now given as:

$$[A]^2 = [\phi][\lambda][\phi]^{-1}[\phi][\lambda][\phi]^{-1} = [\phi][\lambda]^2[\phi]^{-1} \quad (18)$$

Similarly raising $[A]$ to the n^{th} power yields:

$$[A]^n = [\phi][\lambda]^n[\phi]^{-1} \quad (19)$$

Substituting Eq. (16-19) into Eq. (10) gives the implicit solution to the one-dimensional time rate of settlement problem:

$$[u]_n = [\phi][\lambda]^{n-j}[\phi]^{-1}[u]_j \quad \text{for } j = 0, \dots, n \quad (20)$$

Eq. (20) involves multiplication of three matrices, regardless of the number of time increment j . This is true since raising a diagonal matrix to the power n^{th} power is accomplished by raising its diagonal values to the power n . This is precisely the advantage of using Eq. (20) over the conventional finite difference procedure.

It is often convenient to express time in terms of the dimensionless parameter known as the time factor T . Since length of drainage path H_{dp} , the time in question t , and the coefficient of consolidation c_v , we write:

$$T = \frac{c_v t}{H_{dp}^2} \quad (21)$$

The length of drainage path can be expressed in terms of the thickness of the consolidating layer, H_o . For a soil drained at both ends:

$$H_{dp} = \frac{H_o}{2} = \frac{m}{2} \Delta z \quad (22)$$

The coefficient of consolidation can be expressed next in terms of α and the depth and time increments as follows:

$$c_v = \frac{(\Delta z)^2}{\Delta t} \alpha \quad (23)$$

Substituting Eq. (22) and Eq. (23) into Eq. (21) and noting that $t = n\Delta t$ gives the time factor in terms of α , depth and time increments. Thus:

$$T = \frac{\frac{(\Delta z)^2}{\Delta t} \alpha (n \Delta t)}{\left(\frac{m}{2} \Delta z\right)^2} = \frac{4 \alpha n}{m^2} \quad (24)$$

The average degree of consolidation for the entire soil layer at any time U_j is determined in terms of the initial area enclosed by the EPWP versus depth distribution A_o and the area enclosed by the EPWP versus depth at any time A_j . Thus,

$$U_j = 1 - \frac{A_j}{A_o} \quad \text{for } j = 0, \dots, n \quad (25)$$

The areas A_o and A_j may be computed using the numerical methods of integration, such as the trapezoidal rule or Simpson's 1/3 Rule.

4 NUMERICAL EXAMPLE

Determine the average degree of consolidation, time factor, and the EPWP after 5 years for a doubly drained clay layer with a thickness of 18 m, $c_v = 15.0 \text{ m}^2/\text{yr}$, and an initial EPWP distribution of 100 kN/m^2 . Assume $\alpha = 1/6$ and six depth increments.

Solution: Since $H_{dp} = H_o/2 = 9.0 \text{ m}$ and substituting $\alpha = 1/6$ and $\Delta z = 18/6 = 3.0 \text{ m}$ into Eq. (25) gives:

$$\Delta t = \left(\frac{1}{6}\right) \frac{(3)^2}{15} = 0.10 \text{ year} \quad (26)$$

The number of time increments is computed as $n = 5/0.10 = 50$. The EPWP at the boundaries drops to zero at $t > 0$. Furthermore, at $t = 0$ the EPWP $u_{top} = u_{bot} = 100 \text{ kN/m}^2$. This discrepancy between the boundary conditions is resolved by taking the average value of EPWP of 50 kN/m^2 as the initial EPWP at the boundaries. Thus:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}_1 = \begin{bmatrix} 1/6 & 2/3 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 1/6 & 2/3 & 1/6 & 0 & 0 & 0 \\ 0 & 0 & 1/6 & 2/3 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 1/6 & 2/3 & 1/6 & 0 \\ 0 & 0 & 0 & 0 & 1/6 & 2/3 & 1/6 \end{bmatrix} \begin{bmatrix} 50 \\ 100 \\ 100 \\ 100 \\ 100 \\ 100 \\ 50 \end{bmatrix}_0 = \begin{Bmatrix} 91.67 \\ 100.00 \\ 100.00 \\ 100.00 \\ 91.67 \end{Bmatrix}_1 \quad (27)$$

The eigenvalues and eigenvectors for this problem is given below and the $\{u\}_1$ vector can be calculated after 5 years directly without intermediate steps. That is substituting $j = 1$ and $n = 50$ into Eq. (27) gives:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}_{50} = \frac{1}{12} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ \sqrt{3} & -1 & 0 & 1 & -\sqrt{3} \\ 2 & 0 & -1 & 0 & 2 \\ \sqrt{3} & -1 & 0 & -1 & -\sqrt{3} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.9533 & 0 & 0 & 0 & 0 \\ 0 & 5/6 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0.3780 \end{bmatrix}^{49} \begin{bmatrix} 1 & \sqrt{3} & 2 & \sqrt{3} & 1 \\ -3 & -1 & -3 & 0 & 3 \\ 4 & 0 & -4 & 0 & 4 \\ -3 & 3 & 0 & -3 & 3 \\ 1 & -\sqrt{3} & 2 & -\sqrt{3} & 1 \end{bmatrix} \begin{Bmatrix} 91.67 \\ 100.00 \\ 100.00 \\ 100.00 \\ 91.67 \end{Bmatrix}_1 \quad (28)$$

The calculated EPWP vector is given as $\{u\}_{50} = \{0, 6.469, 11.205, 12.938, 11.205, 6.469, 0\}$. The time factor is calculated next by substituting $m = 6$, $\alpha = 1/6$, and $n = 50$ into Eq. (24) which gives $T = 0.926$. The corresponding average degree of consolidation is computed next using Eq. (19) and Simpson's 1/3 rule of integration. That is:

$$U_{50} = 1 - \frac{\left(\frac{3.0}{3}\right)^{[0+4(6.469)+2(11.205)+\dots+0]}}{100(18)} = 91.76\% \quad (29)$$

The time factor and the average degree of consolidation compare rather closely with the analytical values of $T = 0.90$ and $U = 91.20\%$ reported by Perloff and Baron (1976).

5 CONCLUSION

An innovative method for solving the one-dimensional consolidation time rate of settlement numerically is presented. The new technique is efficient and versatile in that it applies to any initial EPWP. Unlike the traditional finite differences techniques, once a model is developed, then arbitrary initial EPWP distribution can be analyzed without the need for reworking the problem. Using the explicit finite difference technique, one would be required to solve a given problem for a particular initial EPWP and the solution is not necessarily applicable to other initial EPWP distributions. This is not the case with the proposed methods presented in this paper. Furthermore, the solution procedure presented herein permits the determination of EPWP at any time without the need for computing intermediate EPWP values. It is also possible to calculate the EPWP at a fraction of a time increment. This eliminates the need for interpolating when intermediate values between increments are needed. This is the case when dealing with the finite difference procedure. This innovative method reduces substantially the roundoff error and computational time associated with other numerical techniques. The solution at any time is given in terms of an initial EPWP vector and is achieved by multiplying three matrices.

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