Integrated Solutions for Infrastructure Development
Edited by Lau, H. H., Tang, F. E., Ng, C. K., and Singh, A.

ISBN: 978-0-9960437-3-1

AN APPLICATION OF HYPERPLASTICITY TO DETERMINE THE BEHAVIOR OF A PLANE STRESS CANTILEVER BEAM

HAI THAN NGUYEN and LAM NGUYEN SY

Faculty of Civil Engineering, Ho Chi Minh City University of Technology, Vietnam

Hyperplasticity is an approach to plasticity theory based on thermodynamic principles. By using this concept, the entire constitutive model behavior can be determined from two scalar potential functions, one is the energy function and the other is the yield function or the dissipation function. In this paper, the hyperplasticity model for homogeneous and isotropic material is derive and implemented in finite element method (FEM) to solve a particular problem in structure analysis, the behavior of a cantilever beam. The Gibbs free energy function and the Von Mises yield function are chosen to derive the elasto-plastic response of structures. Besides, kinematic hardening with single yield surface model is employed to describe the behavior when yield occur. The result shows that the application of hyperplasticity to FEM can be another proper way for non-linear analysis. Moreover, it reveals the ability to develop more sophisticated model using multiple yield surfaces in cases of cyclic loading.

Keywords: Elasto-plastic, Non-linear analysis, Kinematic hardening, Thermodynamic, Yield surface, Flow rule.

1 INTRODUCTION

The principle root of hyperplasticity approach is originally proposed by Ziegler (1977). This work shows that constitutive model of a deformable solid can be defined by using only two scalar functions, the energy function and the dissipation function. Collins and Houlsby (1997) developed an approach to plasticity theory base on thermodynamic for geotechnical materials. Houlsby and Puzrin (2000a and 2000b) continued this work by developing this approach into a consistent and rigorous thermomechanical framework base on generalize thermodynamic for the rate-independent dissipative materials, which can be applied for a wide range of engineering materials.

There are several advantages of hyperplasticity model. One significant feature of this approach is that the derivation of constitutive behavior is established in strict accordance with the first and the second law of thermodynamics; therefore, the hyperplasticity approach guarantees that incremental derivation of plasticity will automatically obey the laws of thermodynamics. Another advanced is the ability to capture behavior under cyclic loading. This is because the past history of loading-unloading process is always reflected clearly through the state of the internal variables.

Various researches have been done by applying the hyperplasticity approach. Nguyen-Sy (2005) employed this concept for the modelling of circular shallow offshore foundation.

Likitlersuang (2005) developed a hyperplasticity model for clay. Nguyen (2005) and Einav (2007) applied this framework for the coupled damage and plasticity models.

In this paper, the hyperplasticity model for a rate-independent kinematic hardening plastic material is introduced and applied to derive behavior of a cantilever metal beam, which is considered as a homogeneous and isotropic material. A comparison between the conventional approach and the hyperplasticity concept is also conducted. Aim of this work is to introduce another rigorous way to simulate the structure behavior, which has enough ability to derive the realistic work from the viewpoint of thermodynamic.

2 THE HYPERPLASTICITY THEORY

The specific formulation of this concept from classical thermodynamic for fluid to solid can be found in Houlsby and Puzrin (2000a and 2000b). This section introduces a brief summary of this concept for small strain continuum mechanic.

The formulation starting by the introduction of an internal state variable in tensor form (α_{ij}). In this paper, the Gibbs free energy function, which take the form $g = g(\sigma_{ij}, \alpha_{ij}, \theta)$, is used. The main goal of the definition of the internal variables is to supply the tools for the study of the dissipation of energy. Formulation from first and second laws lead to the form of dissipation function $d = d(\sigma_{ij}, \alpha_{ij}, \theta, \dot{\alpha}_{ij}) \geq 0$. Tensors $\bar{\chi}_{ij} = -\partial g / \partial \alpha_{ij}$ and $\chi_{ij} = \partial d / \partial \dot{\alpha}_{ij}$ are the generalized stress and the dissipative generalized stress respectively. The fundamental of constitute hypothesis is $\bar{\chi}_{ij} = \chi_{ij}$.

Collins and Houlsby (1997) shows that rate-independent material, which the dissipation is homogeneous first order function, is a degenerate case of Legendre transformation. Therefore, the transformation of and derivative dissipation function give $\dot{\alpha}_{ij} = \lambda \dfrac{\partial y}{\partial \chi_{ij}}$. The role of y is identified with the role of yield surface in conventional approach, but the difference is that it respects to generalized stress χ_{ij} instead of true stress σ_{ij}. Besides, α_{ij} is now identified with the plastic strain.

3 HYPERPLASTICITY FOR SINGLE YIELD SURFACE KINEMATIC HARDENING

In this section, the behavior of model using Von Mises yield criteria and kinematic hardening will be derive. Thermal effect is not considered in this paper; therefore, the dependence on the temperature will not be taken into account.

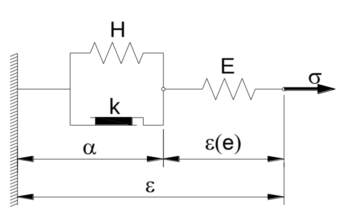

Figure 1. Friction slip and spring model for kinematic hardening.

The mechanical behavior of kinematic hardening with single yield surface in one-dimensional model is expected to be as shown in Figure 1. The hardening parameter H can be seen as the modulus of a spring when yield occur.

The first function specified is the Gibbs free energy function. As Puzrin and Houlsby (2001), the Gibbs free energy function for multi-dimensional model takes the form:

$$g = -\frac{1}{18K}\sigma_{ll}\sigma_{kk} - \frac{1}{4G}\sigma'_{ij}\,\sigma'_{ij} + \frac{H}{2}\alpha_{ij}\alpha_{ij} - \sigma_{ij}\alpha_{ij} \quad (1)$$

Where K is the bulk modulus, G is the shear modulus and H is the hardening parameter of model. From Eq. (1) we can derive:

$$\varepsilon_{ij} = -\frac{\partial g(\sigma_{ij},\alpha_{ij})}{\partial \sigma_{ij}} = \frac{1}{9K}\sigma_{kk}\delta_{ij} + \frac{1}{2G}\sigma'_{ij} + \alpha_{ij} \quad (2)$$

$$-\frac{\partial^2 g}{\partial \sigma_{ij}\partial \sigma_{kl}} = \frac{1}{3K}I^V_{ijkl} + \frac{1}{2G}I^D_{ijkl} = C_{ijkl} \quad (3)$$

In which I^V_{ijkl} , I^D_{ijkl} are tensors that $\sigma_{kk} = I^V_{ijkl}\sigma_{ij}$, $\sigma'_{ij} = I^D_{ijkl}\sigma_{ij}$ and $C_{ijkl} = D^{-1}_{ijkl}$ is the elastic tensor.

The second function is the Von Mises yield function, which takes the form:

$$y^g = \chi'_{ij}\,\chi'_{ij} - 2k^2 \quad (4)$$

Where k is the yield stress in simple shear. From this we can derive as follow:

$$\dot{\alpha}_{ij} = \lambda\frac{\partial y^g}{\partial \chi_{ij}} = 2\lambda I^D_{ijkl}\chi'_{kl} = 2\lambda\chi'_{ij} \quad (5)$$

The multiplier λ can be calculated from Houlsby and Puzrin (2001) and the result $\lambda = \frac{\chi'_{ij}\,\dot{\sigma}_{ij}}{4Hk^2}$ with note that λ is a non-negative multiplier whenever $y = 0$.

In this paper, the increment of stress and strain tensor are not decoupled into deviatoric and volumetric parts, which is different from the formulation as Puzrin and Houlsby (2001). The incremental response is as follow:

$$\dot{\varepsilon}_{ij} = -\frac{\partial^2 g}{\partial \sigma_{ij}\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial^2 g}{\partial \sigma_{ij}\partial \alpha_{kl}}\dot{\alpha}_{kl} = C_{ijkl}\dot{\sigma}_{kl} + \frac{\chi'_{ij}\,\dot{\sigma}_{ij}}{4Hk^2}2\chi'_{kl} = \left(C_{ijkl} + \frac{\chi'_{ij}\,\chi'_{kl}}{2Hk^2}\right)\dot{\sigma}_{kl} \quad (6)$$

The vector notation is needed in order to apply to FEM. In vector notation, the stress component is written as $\sigma = \{\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{yz} \quad \tau_{zx} \quad \tau_{xy}\}^T$. The notation is similar for ε,α,χ . The incremental stress-strain response from Eq. (6) can be rewritten as:

$$\dot{\varepsilon} = C\dot{\sigma} + \frac{(\chi')^T P\dot{\sigma}}{2Hk^2}\chi' = \left(C + \frac{\chi'(\chi')^T P}{2Hk^2}\right)\dot{\sigma} \quad (7)$$

$$\dot{\sigma} = D_{ep}\dot{\varepsilon} = \left(C + \frac{\chi'(\chi')^T P}{2Hk^2}\right)^{-1}\dot{\varepsilon} \quad (8)$$

Where $C=D^{-1}$ is the inverse of elastic matrix and $P=diag\begin{bmatrix}1 & 1 & 1 & 2 & 2 & 2\end{bmatrix}$ is diagonal scaling matrix for the result in vector notation equal to product in tensor notation.

The form of the incremental equation is similar in comparison with conventional approach. However, it is derived by internal state variable and generalized stress instead of true stress.

4 NUMERICAL EXAMPLE

This section explains the result when apply hyperplasticity theory to derive the behavior of a cantilever beam under cyclic load in plane stress problem. See Figure 2 for definition. It should be noted that this is just a theoretical example, so the parameters are assumed base on the properties of steel to view the stress-strain response and the ultimate strength is not considered in this example. A more realistic model needs data from experiments, which requires further researches.

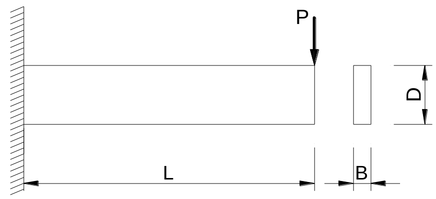

Figure 2. Problem definition.

Geometric parameters: L = 1,000 mm, D = 200 mm, B = 50 mm. Material properties: E = 210,000 MPa, ν = 0.3, H = 30,000 MPa and yield stress σ_P = 240 MPa. There are four loading processes as shown in Table 1.

Table 1. Loading processes.

Step	**Process**			
	1	**2**	**3**	**4**
1	From 40 kN to 176 kN	From 40 kN to 176 kN	From 40 kN to 176 kN	From 40 kN to 176 kN
2	From 176 kN to 56 kN	From 176 kN to -84 kN	From 176 kN to -184 kN	From 176 kN to -184 kN
3	From 56 kN to 216 kN	From -84 kN to 216 kN	From -184 kN to 216 kN	From -184 kN to 188 kN
4				From 188 kN to -200 kN

4.1 The Increment Response of Applied Load and Displacement

In order to check the reasonable of new approach, we have made codes based on the approach of Owen and Hinton (1980) and use the same input data (process 1) to compare the results of load-displacement response at center of free-end section between two approaches as show in Figure 3.

The result shows that the shape of the responses of hyperplasticity model are similar in comparison to the conventional approach. However, there are little difference in values, especially when the yield behavior occur. This is because the meaning of hardening parameter. Owen and Hinton (1980) used the work hardening rule whereas the strain hardening is employed in our work. Despite of the differences, the results show that hyperplasticity is a promising approach. Moreover, the advanced of the new approach is that it can simulate the behavior of cyclic load in three other processes (2, 3 and 4), which cannot be done by the conventional approach. The responses of center point of free-end section in four processes are shown below.

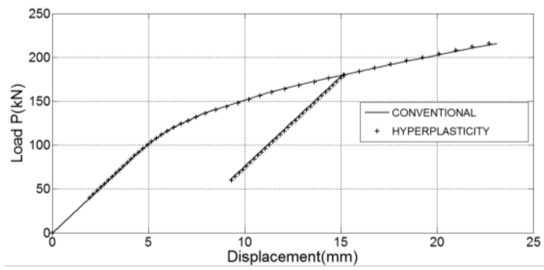

Figure 3. Load-displacement response in comparison with conventional approach.

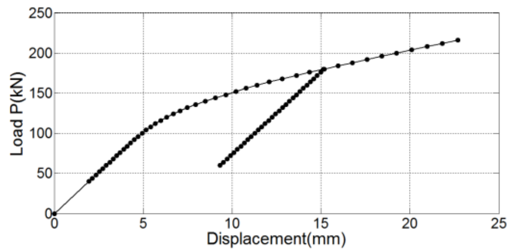

Figure 4. Load-displacement response for process 1.

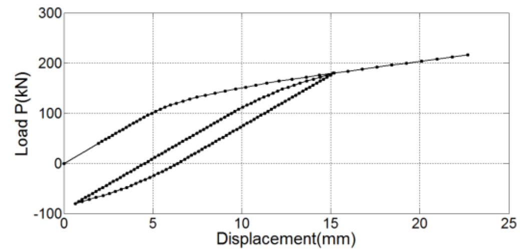

Figure 5. Load-displacement response for process 2.

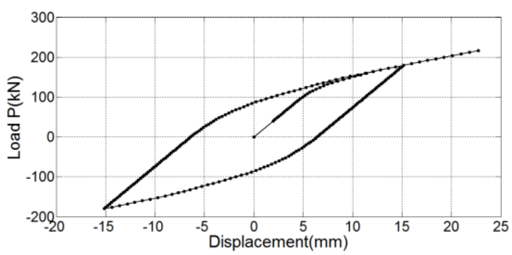

Figure 6. Load-displacement response for process 3.

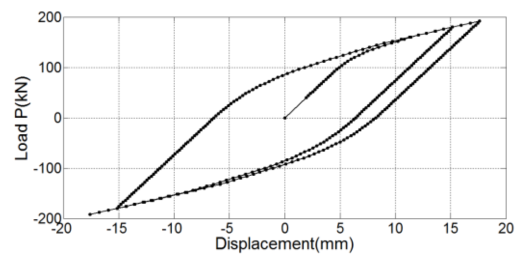

Figure 7. Load-displacement response for process 4.

When the unloading occurs in the first process, the stress state point moves within the yield surface, but it does not touch the yield surface again, so the behavior is elastic and the reloading is elastic too as shown in Figure 4. The second process, see Figure 5, shows a further step in which the stress point touch the yield surface again during unloading. In the third process, which can be seen from Figure 6, the range of unloading step is expanded in order to get over the state at the first time. The fourth process adds another unloading period as shown in Figure 7. It can be seen from the results that if the cyclic processes continue, the load-displacement response will obey the Mashing rule as discussed in Nguyen-Sy (2005).

4.2 The Stress Distribution of the Cantilever Beam

The stress (σ_x) distribution during process 4 is presented at four reference stages: (1) After the first loading increment, when yield behavior has not appeared, (2) at the end of load step 1, (3) at the end of load step 2 and (4) at the end of load step 3.

As shown in Figure 8, the stress distribution looks reasonable. At the first stage, the tensile stresses distribute on top and compressive stresses at bottom of section (the stress on top and at bottom of fixed end section are 126.78 MPa and -126.42 MPa respectively). During the loading period, the distribution areas of tension and compression enlarge to the center of section and to the free end of the cantilever beam as shown at second reference time (maximum stress is 459.25 MPa and minimum stress is -458.14 MPa). The third reference stage is similar to the second but inverse in value due to the change of loading direction (stress on top of fixed end section is -458.70 MPa whereas at bottom is 457.54 MPa). When the reloading is conducted, the distribution at fourth stage is similar to second stage (497.17 MPa and -495.96 MPa on top and at bottom respectively).

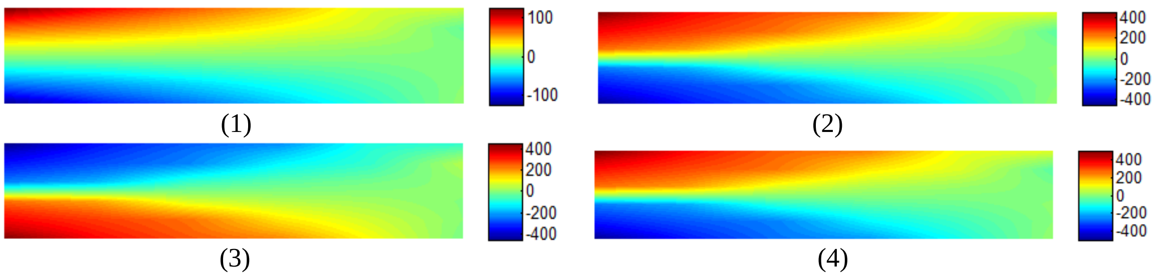

Figure 8. The distribution of σ_x (MPa) during process 4 at four reference stage.

5 CONCLUSION

Applying hyperplasticity theory to nonlinear behavior can be a proper way to simulate the realistic behavior of structure engineering. The presented model can be used to derive the response of structures in various ways of loading process. For a more realistic result, parameter of this approach should be calibrated with experimental data. Besides that, the approach for single yield surface shows the ability to go further to the concept of multiple yield surface, which is more sophisticated model. Hyperplasticty theory can also apply for behavior of various models in structure analysis such as slender beam, pin-jointed structures etc. by appropriate potential functions, which require further studies.

Reference

Collins, I. F., and Houlsby, G. T., Application of thermomechanical principles to the modelling of geotechnical materials. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*. Vol. 453. No. 1964. The Royal Society, 1997.

Einav, I., Houlsby, G. T., and Nguyen, G. D., Coupled damage and plasticity models derived from energy and dissipation potentials. International Journal of Solids and Structures, 44(7), 2487-2508, 2007.

Houlsby, G. T., and Puzrin, A. M., An approach to plasticity based on generalised thermodynamics. *Constitutive Modelling of Granular Materials*. Springer Berlin Heidelberg, 319-331, 2000a.

Houlsby, G. T., and Puzrin, A. M., A thermomechanical framework for constitutive models for rate-independent dissipative materials. *International Journal of Plasticity* 16.9 (2000): 1017-1047, 2000b.

Nguyen, G. D., A thermodynamic approach to constitutive modelling of concrete using damage mechanics and plasticity theory. Oxford: University of Oxford, 2005.

Nguyen-Sy, L., The theoretical modelling of circular shallow foundation for offshore wind turbines. University of Oxford, 2005.

Likitlersuang, S., A hyperplasticity model for clay behaviour: an application to Bangkok clay. University of Oxford, 2003.

Owen, D. R., and Hinton, E., *Finite elements in plasticity, Pineridge Press, UK,* 1980.

Puzrin, A. M., and Houlsby, G. T., Fundamentals of kinematic hardening hyperplasticity. *International journal of solids and structures* 38.21 (2001): 3771-3794, 2001.

Ziegler, H., *An introduction to thermomechanics*, North Holland, Amsterdam, 1977.