CHAPTER II

PRELIMINARIES

2.1 Definitions and Examples

In this section, we introduce precise definitions, notations, and examples which will be used in this research.

Definition 2.1.1. [17] Let S and Γ be non-empty sets. If there exists a mapping $S \times \Gamma \times S \to S$, written $(a, \alpha, b) = a\alpha b$ for all $a, b \in S$ and $\alpha \in \Gamma$, the set S is called a *Γ-semigroup* if S satisfies the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

From associativity relation, we denote $a\alpha b\beta c$ for $(a\alpha b)\beta c$. Then

$$a\alpha b\beta c := (a\alpha b)\beta c = a\alpha(b\beta c).$$

If S is a Γ-semigroup and $\Lambda \subseteq \Gamma$ then S is a Λ-semigroup and in case of $\Lambda = \{\gamma\}$ then S is called a γ-semigroup. .

By Definition 2.1.1, it is obviously that if S is a Γ-semigroup then S is a γ-semigroup for all $\gamma \in \Gamma$.

The following examples are Γ-semigroups.

Example 1. Let S be the set of all $m \times n$ matrices and Γ be the set of all $n \times m$ matrices over a field then for $A, B \in S$ the product AB can not be defined if $m \neq n$ i.e., S is not a semigroup under the usual matrix multiplication. But for all $A, B, C \in S$ and $\alpha, \beta \in \Gamma$ we have $A\alpha B \in S$ and since matrix multiplication is associative, we have $(A\alpha B)\beta C = A\alpha(B\beta C)$. Hence S is a Γ-semigroup.

Example 2. Let X and Y be non-empty sets. If S is the set of all functions from X to Y and Γ is the set of all functions from Y to X then for $f, g \in S$ the composition $f \circ g$ can not be defined if $X \neq Y$. Thus S is not a semigroup under the usual composite function but S is a Γ-semigroup i.e. for all $f, g, h \in S, \alpha, \beta \in \Gamma$ we have $f \circ \alpha \circ g \in S$ and $(f \circ \alpha \circ g) \circ \beta \circ h = f \circ \alpha \circ (g \circ \beta \circ h)$.

Example 3. Let S be the set of all integers of the form $7n + 2$ and Γ be the set of all integers of the form $7n + 5$ where $n \in \mathbb{Z}$. Define

$$a\gamma b := a + \gamma + b$$

for all $a, b \in S, \gamma \in \Gamma$. We will show that S is a Γ-semigroup. Let $a, b \in S$ and $\gamma \in \Gamma$. Then we set $a := 7n_1 + 2, b := 7n_2 + 2$ and $\gamma := 7n_3 + 5$ for some $n_1, n_2, n_3 \in \mathbb{Z}$. Consider

$$a\gamma b = (7n_1 + 2) + (7n_3 + 5) + (7n_2 + 2) = 7(n_1 + n_2 + n_3 + 1) + 2.$$

Thus $a\gamma b \in S$, so S is a Γ-semigroup.

Definition 2.1.2. [24] Let S be a Γ-semigroup. A non-empty set H of S is said to be a *sub Γ-semigroup* of S if $H\Gamma H \subseteq H$.

For $\gamma \in \Gamma$ if we defined $a \circ b = a\gamma b$ for all $a, b \in S$ then S becomes a semigroup and denotes this semigroup by S_γ.

Let $\gamma \in \Gamma$, H is called a *sub γ-semigroup* of S if $H\gamma H \subseteq H$. Then H is called a sub γ-semigroup of S if H is a sub γ-semigroup of S_γ.

Example 4. Let $S = [0, 1]$ and $\Gamma := \{\frac{1}{n} \mid n$ is a positive integer$\}$. Then S is a Γ-semigroup under usual multiplication. Let $M := [0, \frac{1}{k}]$ for all $k \in \mathbb{N}$. We have that M is a non-empty subset of S and $a\gamma b \in M$ for all $a, b \in M$ and $\gamma \in \Gamma$. Thus M is a sub Γ-semigroup of S.

Definition 2.1.3. [24] Let S be a Γ-semigroup. An element $a \in S$ is *regular* if $a = a\alpha x\beta a$ for some $x \in S$ and for some $\alpha, \beta \in \Gamma$.

S is a *regular Γ-semigroup* if for all elements of S are regular.

Example 5. Let $\mathbb{Q}^-$ be the set of all negative rational numbers and $\Gamma := \{-\frac{1}{p} \mid p$ is prime$\}$. Then $\mathbb{Q}^-$ is a Γ-semigroup under usual product of rational numbers.

Next, we will show that $\mathbb{Q}^-$ is a regular Γ-semigroup.

Let $a := -\frac{m}{n} \in \mathbb{Q}^-$ for $m, n \in \mathbb{N}$. Then we can write $m = p_1 p_2 \cdots p_k$ where p_i are prime for all $i = 1, 2, ..., k$ and $k \in \mathbb{N}$. Choose $\alpha := -\frac{1}{p_1}$ and $\beta := -\frac{1}{p_k}$ are elements in Γ.

Taking $x := -\frac{n}{p_2 p_3 \cdots p_{k-1}} \in \mathbb{Q}^-$, we have that

$$\begin{aligned} a\alpha x\beta a &= (-\frac{p_1 p_2 \cdots p_k}{n})(-\frac{1}{p_1})(-\frac{n}{p_2 p_3 \cdots p_{k-1}})(-\frac{1}{p_k})(-\frac{p_1 p_2 \cdots p_k}{n}) \\ &= -\frac{p_1 p_2 \cdots p_k}{n} \\ &= a. \end{aligned}$$

Thus a is regular which implies that $\mathbb{Q}^-$ is a regular Γ-semigroup.

Definition 2.1.4. [25] Let S be a Γ-semigroup and $a \in S$. Let $x \in S$ and $\alpha, \beta \in \Gamma$. An element x is said to be an *(α, β)-inverse of a* if $a = a\alpha x\beta a$ and $x = x\beta a\alpha x$.

$V_\alpha^\beta(a)$ denotes the set of all (α, β)-inverses of a, i.e.

$$V_\alpha^\beta(a) := \{x \in S \mid x = x\beta a\alpha x, a = a\alpha x\beta a\}.$$

Example 6. By Example 5, we have

$$\begin{aligned} x\beta a\alpha x &= (-\frac{n}{p_2 p_3 \cdots p_{k-1}})(-\frac{1}{p_k})(-\frac{p_1 p_2 \cdots p_k}{n})(-\frac{1}{p_1})(-\frac{n}{p_2 p_3 \cdots p_{k-1}}) \\ &= -\frac{n}{p_2 p_3 \cdots p_{k-1}} \\ &= x. \end{aligned}$$

Thus $-\frac{n}{p_2 p_3 \cdots p_{k-1}} \in V_{(-\frac{1}{p_1})}^{(-\frac{1}{p_k})}(a)$.

Definition 2.1.5. [19] A regular Γ-semigroup S is called an *inverse Γ-semigroup* if for all $a \in S, \alpha, \beta \in \Gamma$, $V_\alpha^\beta(a) \neq \emptyset$ implies $\mid V_\alpha^\beta(a) \mid = 1$.

That is, every element a of S has a unique (α, β)-inverse.

Definition 2.1.6. [16] Let S be a Γ-semigroup. An element $e \in S$ is said to be an *α-idempotent* of S, where $\alpha \in \Gamma$ if $e\alpha e = e$. We denote the set of all α-idempotents of S as follows : $E_\alpha(S) := \{e \in S \mid e\alpha e = e\}$.

Let $E(S) := \bigcup_{\alpha\in\Gamma} E_\alpha(S)$. Then we called $E(S)$ the set of all idempotents of S. If x is an idempotent of S, we mean $x \in E(S)$.

Definition 2.1.7. A regular Γ-semigroup S with the set of all idempotents $E(S)$ will be called a *locally inverse* Γ*-semigroup* if $e\Gamma S\Gamma e$ is an inverse Γ-semigroup for every $e \in E(S)$.

Definition 2.1.8. [26] The Green's equivalence relation $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ on a Γ-semigroup S are defined by the following rules :

(1) $a\mathcal{L}b$ if and only if $S\Gamma a \cup \{a\} = S\Gamma b \cup \{b\}$.

(2) $a\mathcal{R}b$ if and only if $a\Gamma S \cup \{a\} = b\Gamma S \cup \{b\}$.

(3) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

(4) $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$.

The $\mathcal{L}$-class (resp. $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class) containing the element a will be written L_a (resp. R_a, H_a, D_a).

2.2 Basic Properties

In this section certain basic results are presented. Reference will be made to these throughout the thesis.

Proposition 2.2.1. [25] *S is a regular Γ-semigroup if and only if $V_\alpha^\beta(a) \neq \emptyset$ for all $a \in S$, for some $\alpha, \beta \in \Gamma$.*

Theorem 2.2.2. [21] *Let S be a Γ-semigroup, S is an inverse Γ-semigroup if and only if*

(1) *S is regular,*

(2) *if $e, f \in E_\alpha(S)$ then $e\alpha f = f\alpha e$ where $\alpha \in \Gamma$.*

Lemma 2.2.3. [26] *Let S be a Γ-semigroup. Then for all $a, b \in S$, we have*

(1) *$a\mathcal{L}b$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$.*

(2) $a\mathcal{R}b$ *if and only if* $a = b$ *or there exist* $x, y \in S$ *and* $\alpha, \beta \in \Gamma$ *such that* $a = b\alpha x$ *and* $b = a\beta y$.

(3) $a\mathcal{H}b$ *if and only if* $a\mathcal{L}b$ *and* $a\mathcal{R}b$.

(4) $a\mathcal{D}b$ *if and only if there exists* $c \in S$ *such that* $a\mathcal{L}c$ *and* $c\mathcal{R}b$.

Lemma 2.2.4. [26] *Let* S *be a* Γ*-semigroup,* $\alpha \in \Gamma$ *and* e *be an* α*-idempotent. Then*

(1) $a\alpha e = a$ *for all* $a \in L_e$.

(2) $e\alpha a = a$ *for all* $a \in R_e$.

(3) $a\alpha e = a = e\alpha a$ *for all* $a \in H_e$.

Proposition 2.2.5. *Let* S *be a regular* Γ*-semigroup. Then for all* $a \in S$*, there exist* $\alpha, \beta \in \Gamma$ *and* $a' \in V_\alpha^\beta(a)$ *such that* $a'\beta a \in E_\alpha(S)$ *and* $a\alpha a' \in E_\beta(S)$.

For a Γ-semigroup S and $\alpha \in \Gamma$, we define a new operation $\circ$ on S by

$$a \circ b := a\alpha b \quad \text{for all} \quad a, b \in S.$$

Then $(S, \circ)$ is a semigroup. Such semigroup is denoted by S_α. A nonempty set T is called a *subgroup* [27] of S_α if $(T, \circ)$ is a group.

Definition 2.2.6. [21] A Γ- semigroup S is called *left (right) simple* if it has no proper left (right) ideals. S is said to be *simple* if it has no proper ideals.

Theorem 2.2.7. [21] *Let* S *be a* Γ*- semigroup.* S_γ *is a group for some* $\gamma \in \Gamma$ *if and only if* S *is both left simple and right simple.*

Definition 2.2.8. Let S be a Γ-semigroup without zero. We shall say that S is *completely simple* if S is simple and if it contains a primitive idempotent.

Corollary 2.2.9. [24] *Let* S *be a* Γ*-semigroup. If* S_α *is a group for some* $\alpha \in \Gamma$ *then* S_α *is a group for all* $\alpha \in \Gamma$.

Definition 2.2.10. [28] Let S be a Γ-semigroup and $a \in S$. If $a = a\alpha x\beta a$ and $a\alpha x = x\beta a$ for some $x \in S, \alpha, \beta \in \Gamma$ then an element a is called an *(α, β)-completely* *regular element* of S.

A Γ-semigroup S will be called *completely regular* if every element of S is an (α, β)-completely regular element for some $\alpha, \beta \in \Gamma$.

Corollary 2.2.11. [28] *Let S be a Γ-semigroup and $e \in E(S)$. Then H_e is a subgroup of S_γ for some $\gamma \in \Gamma$.*

Theorem 2.2.12. [28] *Let S be a Γ-semigroup. Then the following statements are equivalent.*

(1) *S is completely regular.*

(2) *Each element of S lies in a subgroup of S_γ for some $\gamma \in \Gamma$.*

(3) *Every H-class is a subgroup of S_γ for some $\gamma \in \Gamma$.*

Definition 2.2.13. [3] Let X be a partially ordered set. An equivalence relation ρ on X is called *strictly compatible with* $\leqslant$ if no two distinct ρ-related elements are comparable with respect to $\leqslant$.