DEVELOPMENT OF AN INQUIRY-BASED LEARNING UNIT TO PROMOTE HIGH-SCHOOL STUDENTS' CONCEPTUAL UNDERSTANDING OF EARTH GEOMETRY AND TRIGONOMETRY

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ABSTRACT

In order to make the Earth, as represented by the globe and the map, more understandable to high-school students, an inquiry-based learning unit on "The Earth and Trigonometry" was developed to supplement the traditional presentation of the longitude, latitude, the great circle, the meridian, and the dateline. Several pilot studies were conducted in other schools before the teacher arrived at the final instructional unit reported here. Teachers' observation, students' comments, and experts' recommendations were discussed and analyzed to modify the learning unit, physical models, and illustrations for better understanding of the geographical parameters above. Students from two schools participated in the developed learning unit as groups in which they asked and answered questions, discussed with groupmates, and class. The teacher acted as the facilitator who guided the students through the objectives by scaffolding their learning with basic trigonometry and geometry that they had learned from previous exposures. The teacher also conducted debriefing at which students and teacher participated to achieve a better outcome overall. Students' conceptual understanding from pre-, post-tests, observation of their co-operation, and their feedbacks to the attitudinal questionnaire were analyzed. It was found that the most difficult topic was the 3-D projection on the three orthogonal axes; x, y, z. However, using improved three-dimensional physical models and illustrations, better conceptual understanding to the longitude, latitude, the great circle, and its distance was achieved.

KEY WORDS: EARTH GEOMETRY / GEOGRAPHICAL PARAMATERS / GUIDED INQUIRY-BASED LEARNING UNIT / HIGH-SCHOOL STUDENTS / PHYSICAL MODEL / TRIGONOMETRY

120 pages

การพัฒนาหน่วยการเรียนแบบสืบเสาะหาความรู้ สำหรับนักเรียนมัธยมศึกษาตอนปลาย เพื่อพัฒนามโนทัศน์เรื่อง โลกของเราและตรีโกณมิติ

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บทคัดย่อ

งานวิจัยนี้ได้พัฒนาหน่วยการเรียนแบบสืบเสาะหาความรู้ สำหรับนักเรียนมัธยมศึกษาตอนปลาย เรื่อง โลกของเราและตรีโกณมิติ เพื่อให้นักเรียนเข้าใจแง่มุมต่างๆเกี่ยวกับโลกโดยอาศัยความรู้ของตรีโกณมิติและเรขาคณิตเบื้องต้น โดยได้พัฒนาสื่อการเรียนรู้จำลองโลกและแผนที่โลก หน่วยการเรียนรู้นี้เสริมความเข้าใจในเรื่องภูมิศาสตร์ เรื่อง เส้นแวง เส้นรุ้ง เส้นgreat circle เส้นแวงที่ศูนย์องศา และเส้นแบ่งเวลา หน่วยการเรียนรู้นี้ได้ทดลองใช้กับนักเรียนชั้นมัธยมศึกษาปีที่ 5 จากโรงเรียนสหศึกษา 2 โรงเรียน จำนวน 62 คน โดยหน่วยการเรียนรู้นี้ได้ออกแบบมาเพื่อให้โอกาสนักเรียนได้ถาม ตอบ และอภิปรายกับเพื่อนทั้งในกลุ่มและต่างกลุ่ม โดยครูทำหน้าที่คอยช่วยเหลือ ไม่ว่าจะเป็นการใช้คำถามกระตุ้นให้นักเรียนคิดหรืออภิปรายสรุปความคิดรวบยอดกับนักเรียนทั้งห้องหลังจากทำกิจกรรมในแต่ครั้ง และใช้กระบวนการที่ผสมผสานระหว่างวิธีเชิงคุณภาพและวิธีเชิงปริมาณในการเก็บข้อมูล ได้แก่ แบบทดสอบความคิดรวบยอดก่อนและหลังเรียน การสังเกตการทำงานเป็นกลุ่มของนักเรียน ผลงานของนักเรียน และแบบสอบถาม จากการศึกษาพบว่า บทเรียนนี้ช่วยให้ผู้เรียนสามารถพัฒนามโนทัศน์เรื่อง โลกของเราและตรีโกณมิติ และพบว่า การฉายภาพจากจุดบนพื้นผิวโลก (ทราบค่ามุมของเส้นรุ้งและเส้นแวง) ไปยังระบบพิกัดฉากสามมิติ เป็นเรื่องที่ยากสำหรับนักเรียน ดังนั้น เพื่อส่งเสริมความเข้าใจของนักเรียน ผู้วิจัยแนะนำให้พัฒนาแบบจำลองสามมิติขึ้นใหม่ โดยดัดแปลงจากที่ได้ใช้มาแล้วเล็กน้อย นอกจากนี้ผู้เรียนยังมีเจตคติที่ดีต่อการเรียนรู้แบบสืบเสาะหาความรู้ที่เน้นการบูรณาการระหว่างวิชา การใช้สื่อการเรียนรู้ (แบบจำลองลูกโลกและรูปวาด) และการทำงานร่วมกับผู้อื่น ดังแสดงจากการเขียนประเมินตนเอง ทั้งความพึงพอใจของนักเรียนต่อการจัดการเรียนรู้เรื่องโลกของเราและตรีโกณมิติ และความคิดเห็นของนักเรียนต่อการเรียนเรื่องโลกของเราและตรีโกณมิติ และจากการที่ได้รับประสบการณ์การเรียนแบบบูรณาการความรู้ จึงส่งผลต่อการเปลี่ยนแปลงทัศนคติของนักเรียนต่อการเรียนวิชาคณิตศาสตร์ในทางที่ดีขึ้น

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CHAPTER I
INTRODUCTION

Overview

This chapter introduces the background information and the rationale for this study. This part includes objectives, research questions, and significance of the research study. The definitions of terms relevant to the study are described. The outline of the remaining chapters is also included at the end of this part.

1.1 Background and Justification for Research

Secondary students, especially the upper secondary ones, know terms such as equator, longitude, latitude, international dateline, prime meridian and the numbers (in degree) designating them. However, when asked how the numbers in degree come about, most of them have no idea. Although having been familiarized with various projections from their geography classes, most students cannot begin to work out the distance between two points on the map even when the longitude and latitude numbers are given (personal communication). They generally do not connect their views of countries on a 3-D globe with those on a flattened map, especially, the Mercator Projection, which has parallel lines, for both longitudes and latitudes, cutting perpendicularly on a rectangular surface. As a result they cannot see the distortion in terms of size and shape of lands and waters in the projection. We attribute the problems to their lack of a firm background on trigonometry and thus are not able to apply knowledge to real world situations.

Based on our observation, the above problems are prevalent in spite of students' having been exposed to a rather high level of trigonometry and geometry, which are generally taught without any mention of application in daily life, e. g., measurement of irregular-shaped land, let alone in physics and geography. We agree with previous reports (Kilpatrick, Swafford & Findell, 2001; Gutstein, 2007b) that the mathematical literacy developed through the exploration of real world problems allows students to build their mathematical proficiency including their ability to manage and set constraints. For teachers then, the challenge is often the identification and development of real world problems that engage and motivate students. Navigation and geography can be one context of interest to students in their perception of usefulness of trigonometry (Faulkner, 2004) and hence more enthusiasm in learning the latter. Concerning geography, particularly earth geometry, when asked why the airline travel map commonly shows curved lines (even when there are only a few lines, i. e., no crowding of lines) linking cities of their service, we constantly receive the unexpected answer that the lines are curved because planes have to take off (line curving up) and land (line curving down). None has mentioned the great circle as the shortest route along which pilots fly from one airport to the next.

Trigonometry as it is taught in secondary school is concerned with the ratios of lengths of the sides in or the relationships between the sides and the angles of a right triangle. A unit circle is also used to explain the periodicity of trigonometric functions. Together, they are used to introduce students to basic trigonometric functions in a trigonometry course. However, learning trigonometric ideas is difficult for students and the causes of the difficulties seem to be multifaceted and interrelated. First, trigonometric functions are perhaps students' first encounter with operations that cannot be evaluated algebraically, the kinds of operations about which they have trouble reasoning (Weber, 2005). Those introduced to the subject via triangle trigonometry, which is more comprehensible than circle trigonometry at the early stage of learning (Kendal & Stacey 1997), have (i) to relate triangular pictures to numerical relationships, (ii) to cope with trigonometric ratios, and (iii) to manipulate the symbols involved in such relationships (Blackett & Tall, 1991). To facilitate the memorization of these ratios, students are often taught the mnemonic *SOHCAHTOA* with the detrimental effect that they stop trying to make sense of the work because they have a simple rule to follow (Cavanagh, 2008). Teaching triangle trigonometry before circle trigonometry also leads to students' understanding of trigonometric functions as taking right triangles, not angle measures, as their arguments (Thompson, 2008). In fact, students never develop a coherent concept of angle measure. To a certain extent, the same can be said about teachers (Thompson, Carlson, & Silverman, 2007). As such, both teachers and students have trouble transferring to circle trigonometry (Bressoud, 2010; Thompson, 2008), which is the foundation for more advanced topics in science and mathematics, as evidenced by the fact that they have to rely on yet another mnemonic, *All Students Take Calculus*, to determine the signs of trigonometric functions in different quadrants (Brown, 2005).

Some of the studies cited above also suggested remedies for the problems of their concern. Blackett & Tall (1991) employed a computer program that draws the desired right triangles to facilitate students' exploration of the relationship between numerical and geometric data. Quinlan (2004) and Cavanagh (2008) engaged students with hands-on activities involving tangent ratios before formally introducing trigonometric functions. As echoed by Bressoud (2010), Weber (2005) introduced circle trigonometry before triangle trigonometry, asked students to estimate the results of trigonometric functions, and posed questions that required them to reason about these functions. Thompson, Carlson, & Silverman (2007) used probing questions to help teachers realize the necessity of measuring an angle by its subtending arc length, had them use a string having the same length as the radius to measure angles and estimate sine and cosine, and used another set of questions to help them see that this angle measure leads to a coherent system of meanings in trigonometry.

For most students to grasp certain aspects of the globe well one needs to bring in basic trigonometry and a little non-Euclidean geometry together with manipulatives such as analogous physical objects and illustrations. McNeil and Jarvin (2007), defined "*manipulatives*" as concrete objects used to help students understand abstract concepts in the domain of mathematics. This is supported by Thompson (1994), who believed that concrete materials can provide something on which students can act. Furner, Yahya and Duffy (2005, p. 17) suggested that, "*the use of manipulatives provides teachers with great potential to use their creativity to do further work on mathematics concepts as an alternative to merely relying on worksheets. Consequently, students are learning mathematics in an enjoyable way, making connections between the concrete and the abstract*".

Various educational approaches have been employed for teaching and learning mathematics. Some researchers explored the power of meaningful learning in which good ideas, concepts or skills in instruction co-exist in a context which emphasizes connections with other subjects (Grouws & Cebulla, 2000). This way students should be motivated to construct their own understanding of mathematical concepts. The selection of the appropriate approaches in the appropriate circumstances would smooth the way for students to focus on the larger goal of solving the applied problems. To that end, the inquiry-based approach has been proposed to encourage students not only to investigate, share, and explain their ideas but also to seek ways to apply their knowledge to solve real world problems as suggested by Jarrett (1997), Hinrichsen and Jarrett(1999), Gialamas, Cherif, Keller and Hansen (2000), Carter, (2004), and Fosnot (2003). Thus to solve the above problems by linking earth geometry with trigonometry, we developed the learning unit called "the Earth and trigonometry" in which three hand-held models and line drawings were used as analogous physical and visual objects. The inquiry-based learning unit was implemented (piloted) to 250 upper secondary students in 5 schools successively while the models, illustrations instructional details and assessment instruments were modified. Here we report our findings of latest research study with some comments on our previous experiences.

1.2 Objectives of the Research Study

The principal objectives of this study are as follows:

1) To develop a mathematical learning unit based on inquiry approach for Thai high-school students, namely "The Earth and Trigonometry".
2) To use the lesson learned from the pilot study for modification of the learning unit.
3) To evaluate the effectiveness of the developed learning unit on conceptual understanding of the Earth by applying basic trigonometry and geometry.
4) To investigate for students' perception toward the newly developed learning unit.

1.3 Research Questions

To evaluate the effectiveness of the developed instruction, we seek the answers to the following questions:

1) To what extent can the developed learning unit promote high-school students’ conceptual understanding of distances on the Earth surface by applying basic trigonometry and geometry?
2) What is the students’ perception toward the developed learning unit?

1.4 Significance of the Research Study

The inquiry approach is employed to construct a new mathematical learning unit for high-school students. The developed learning unit should not only support students’ conceptual understanding of the Earth and trigonometry in order to solve the shortest distance (between points) on the spherical Earth surface, but can also integrate mathematical concepts (i.e. trigonometry and geometry) and connect mathematics with the other discipline (i.e. geography) in real life contexts. The significance of this research study is in using real-world problems together with an appropriate learning process and manipulatives in the instructional unit to enhance students’ achievement.

1.5 Definition of Terms

Students’ conceptual understanding of basic trigonometry means students are able to determine the trigonometric functions such as sine, cosine and tangent, and calculate the length of a missing side of a right angle triangle and a unit circle.

Students’ conceptual understanding of latitude and longitude means students are able to compare the arc length between points along the circles of the latitude and longitude.

Students' conceptual understanding of applied trigonometry means students are able to calculate the length of the missing side of a right triangle embedded in a circle and explore the concept of the radian. They were required to calculate the arc length of a sector given the angles at the centre.

Students' conceptual understanding of the great circle distance on the Earth means students are able to determine the Cartesian coordinates system in 3D using the angled of latitude and longitude of the Earth and right angled triangle in trigonometry. Moreover, students can find the shortest distance (between points) on the spherical Earth surface.

Inquiry-based approach means a teaching and learning strategy involving questions generated by teachers and students. Teachers are facilitators who provide students with questions and challenging activities. Students are allowed to investigate and construct their own understanding by taking active roles in class, discussing their finding, and sharing their ideas with their groups and with the whole class.

Guided-inquiry activity means a set of learning activity based on the inquiry approach. Teachers not only design the activity by posing the questions but also provide needed information for students to do the activity, i. e., the physical models and illustrations. Students learn new content by finding the answers and sharing their results with peers.

Inquiry-based mathematical learning unit means the mathematical learning unit with authentic problems created by the researcher and some members of the committee. The problems involve the shortest distance between two cities on the spherical Earth surface. The inquiry-based learning unit is also based on guided-inquiry activities. Students are required to work cooperatively with their groupmates and brainstorm to identify the mathematical concepts in order to solve such problems.

1.6 Organization of the Thesis

This study designs a new trigonometry learning unit, “The Earth and Trigonometry”, based on inquiry approach to promote high-school students’ conceptual understanding of Earth geometry and trigonometry. Students use their mathematical knowledge to solve the real world problems of finding the shortest distance between two cities on the spherical Earth surface. This research is organized into six chapters as follows:

Chapter One provides background and the rationale for investigation of this study. This part also specifies the objectives of the research study, followed by the research questions, the significance of the research project, and the definitions of terms relevant to this study.

Chapter Two provides literature reviews related to the scope of this study. The mathematical review involving applied problems in trigonometry for high-school students are described. The detail of the geography of the Earth concerning latitude, longitude, and the great circle are provided. This chapter also describes the use of physical models and illustrations based on inquiry approach to mathematics education.

Chapter Three presents methodology and method employed in the studies on development of a newly mathematical learning unit on “The Earth and Trigonometry” based on inquiry approach for high-school students. The pilot study and the participants were informed. The implementation of learning units including three activities and the procedures used for data collection and data analysis (e. g., the conceptual understanding test, the worksheet, students’ reflection and students’ perception) were also described.

Chapter Four reports the research findings on the pilot study and on the effectiveness of the latest modified learning unit based on inquiry approach to promote student conceptual understanding of Earth geometry and trigonometry. The results include students’ reflection on their understanding and students’ perception on the learning unit.

Chapter Five discusses the success of the learning unit in promoting students’ conceptual understanding. This chapter emphasizes the impact of the developed learning unit based on inquiry approach on student outcomes. The advantages and disadvantages of the learning unit was discussed based on the quantitative results and students’ reflection.

Chapter Six presents the conclusion of the overall findings of the research study. There were also related to the recommendations of this study. This chapter also provides the limitation of this study and recommendations for future studies and further development.

CHAPTER II
LITERATURE REVIEW

Overview

This chapter presents the mathematical review, the detail of geography of the Earth, and educational review. An application of trigonometry to real world problems for high-school students and basic trigonometry in a right triangle and a unit circle including the components of a circle are described. A description of the Earth which locates points on the Earth with latitude and longitude, and the great circle is also provided. A review on the educational aspects is divided into two parts. The first part describes an inquiry approach for teaching and learning science and mathematics. The other part provides the effectiveness of physical models and illustrations. The theoretical framework of education research for this study is at the end of the chapter.

Mathematical Review

Applied mathematics is an important element for students to develop understanding of the utility of mathematics in society. In countries around the world, it is expected that students will not only be able to think about trigonometric functions, but also be able to apply this knowledge to real world situations. These applications are meant to help students appreciate the power of mathematics and understand its relevance to their future careers. In addition, mathematics is required in the work place and so it is an essential ingredient before high school students graduated (Steen, 1989).

The Trends in International Mathematics and Science Study (TIMSS) in 2009 is an effective international assessment around the world, for example, United States and Thailand. TIMSS provided the “applying” (p. 10) as a part of the cognitive domain on mathematics assessment that focuses on students’ understanding and applying their knowledge to solve problems. Moreover, students can use applied mathematics to develop and create a new knowledge or technology in order to solve real world problems. Along the TIMSS scores in 2007, they showed the cognitive domain scores of eighth-grade students, by country in which the Thai students' average score was lower than the TIMSS scale average. Thai eighth-graders' scores in the *applying* domains were 54 scale score points below the TIMSS scale score average of 500.

The curriculum is developed to support student learning of concepts. Standards are designed to provide a framework for instruction in schools. Teachers use both curriculum and standards to achieve learning goals with students. Countries around the world generate policy documents like standards to support teachers in their work. For example, In Thailand, The Ministry of Education created the standards document, Basic Education Curriculum (BEC), as the core curriculum for national education for preparing school curriculums. The BEC encourages the "decentralisation of educational authority", allowing local districts to generate their own curricular materials and activities. This more allowed various groups and including local educators to develop and disseminate standards and curricular. Applied trigonometry concerned with three area of mathematics (e.g. measurement, geometry, and mathematical skills and process) that The Ministry of Education (2008) prescribed. One of the aims of Basic Education Core Curriculum 2008, students can apply trigonometric ratio as the concepts of relation and function to solve various situations and they have attained creative thinking's ability.

The Ministry of Education (2008) favored geography as one of the main necessary strands to learn social studies, religion and culture. They prescribed student's understanding of physical characteristics of the Earth, for example, the students in grade 5 should be able to know positions (geographical specifications, latitude, and longitude), distance and direction of their own region. Historically trigonometry was used in navigation and astronomy. For example, in navigation, trigonometry was employed to find the distance between two cities on the Earth surface. In astronomy, trigonometry was used to estimate the distance between two stars.

National Council of Teachers of Mathematics (NCTM) has established mathematics standards in the areas of curriculum, teaching, and assessment. Based on the Standards for School Mathematics: Prekindergarten through Grade 12, mathematical processes refer to five process skills that include problem solving, reasoning and proof, communication, connections, and representation (NCTM, 2000). They recognize and apply mathematics in contexts outside of mathematics. School mathematics experiences at all levels should include opportunities to learn about mathematics by working on problems arising in contexts outside of mathematics. These connections can be to other subject areas and disciplines as well as to students' daily lives, especially, students in grades 9–12 students should be confidently using mathematics to explain complex applications in the outside world.

In addition, The Australian mathematics curriculum ensures that the links between the various components of mathematics, and to other disciplines, are made clear (ACARA, 2010). Mathematics is composed of multiple but interrelated and interdependent concepts and systems which students apply in other disciplines. The curriculum is written with the expectation that schools will ensure that all students benefit from access to the power of mathematical reasoning and be able to apply their mathematical understanding creatively and efficiently. The mathematics curriculum provides students with carefully paced, in-depth study of critical skills and concepts. It encourages teachers to facilitate students to become self-motivated, confident learners through inquiry and active participation in challenging and engaging experiences.

All of these above introduced applied mathematics to real world situations for high school students. They inspired us to explore the ideas for the interesting questions are as follows: “Are future citizens of the world in form of high-school graduates ready to respond to the problems that arise in the world?” and “What knowledge is necessary for students to solve real world problems that involve mathematics?”

2.1 The Instruction of Trigonometry

Trigonometry is essential for the study of several topics in mathematics, for example, vector, complex number, and several coordinate systems. In addition, trigonometric functions are used to model periodic phenomena, both in physical science and in life science, as well as to analyze force and motion, among other things. However, as pointed out by Blackett & Tall (1991), *the initial stages of learning the* *ideas of trigonometry are fraught with difficulties*. They focused on the difficulty arising from the visualization of the right triangle pertaining to the problem. As a solution, they employed a computer program that draws the desired right triangles to facilitate students' exploration of the relationship between numerical and geometric data, with significantly positive results, especially for female participants. Weber (2005) argued that trigonometric functions are perhaps students' first encounter with operations that cannot be evaluated algebraically, the kinds of operations about which they have trouble reasoning. So he tried to develop their feel for the functions by asking them to estimate the results of trigonometric functions and posing questions that required them to reason about these functions. Nevertheless, the more, perhaps most, important cause of the difficulties has to do with the two distinct conceptual approaches to trigonometry, namely, right-triangle trigonometry and unit-circle trigonometry, as evidenced by the fact that the majority of recent research studies concentrated on this issue. Even though triangle trigonometry is more comprehensible than circle trigonometry at the early stage of learning (Kendal & Stacey 1997), those introduced to the subject via triangle trigonometry have trouble transferring to circle trigonometry (Bressoud, 2010; Thompson, 2008). Since most teachers also experience this instructional sequence, they are no exception (Thompson, Carlson, & Silverman, 2007). In triangle trigonometry, a trigonometric function is defined as a ratio between sides of a right triangle with one of the two acute angles being its argument. When having to evaluate a trigonometric function whose argument is an obtuse (or greater /negative) angle, students usually do not know how to draw the referent triangle. On the other hand, in circle trigonometry, the argument of a trigonometric function is the central angle, measured counterclockwise from the x-axis, of the circular sector belonging to the unit circle centered at the origin. As such, it can take on any real value. In addition, students are used to angles being measured in degrees, the common unit in triangle trigonometry, whereas in circle trigonometry angles are usually measures in radians. The two units differ not only in their scales, but also in the ways their definitions are put forward. When measuring an angle in degrees, one generally refers to the amount of turn or the openness of the angle, whereas the measurement of an angle in radians refers to the arc length subtending the central angle and takes the radius as the unit of measurement. Thompson (2008) and Thompson, Carlson, &

Silverman (2007) argued that only by understanding the latter definition of an angle measure can one develop a coherent system of knowledge in trigonometry, the task accomplished by neither students nor teachers.

With the aforementioned conceptual changes required to transfer from triangle trigonometry to circle trigonometry, the biggest obstacle to some students may be a mental one. To facilitate the memorization of these ratios, they are often taught the mnemonic *SOHCAHTOA*—*S*ine is *O*pposite over *H*ypotenuse, *C*osine is *A*djacent over *H*ypotenuse, *T*angent is *O*pposite over *A*djacent— with the detrimental effect that they stop trying to make sense of the work because they have a simple rule to follow (Cavanagh, 2008). As a result, they are so used to thinking in terms of right triangles that they conceptually regard trigonometric functions as taking right triangles, not angle measures, as their arguments (Thompson, 2008). In their quest to help teachers develop coherent mathematical meanings, Thompson, Carlson, & Silverman (2007) used probing questions to elicit the realization of the necessity of measuring an angle by its subtending arc length from teachers, had them use a string having the same length as the radius to measure angles and estimate sine and cosine, and used another set of questions to help them see that this angle measure leads to a coherent system of meanings in trigonometry.

Trigonometry as it is taught in Thai secondary school is concerned with the ratios of lengths of the sides in or the relationships between the sides and the angles of a right triangle. A unit circle is also used to explain the periodicity of trigonometric functions. Together, they are used to introduce students to basic trigonometric functions, i. e. sine, cosine, and tangent, in a trigonometry course. In addition, trigonometric identities are used to simplify expressions of trigonometric functions. The important thing is for students to be able to apply trigonometry to real world problems.

2.1.1 The ratio approach

In the ratio approach, the trigonometric functions are defined using the ratios of lengths of the sides in a right triangle or the relationship between the angle and the length of one side of a given right triangle, for example, $\sin\theta = \frac{y}{z}$, $\cos\theta = \frac{x}{z}$, and $\tan\theta = \frac{y}{x}$ (Figure 2.1), where theta is between zero and ninety degrees and x, y, z are the sides of a right triangle.

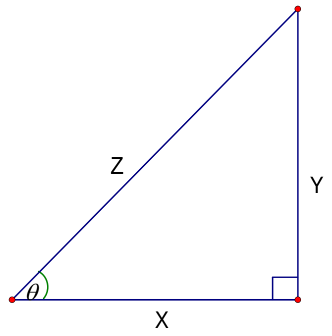


Figure 2.1 Example of the right triangle

2.1.2 The unit circle approach

The unit circle is a circle with a radius of one and centered at the origin (0, 0) in a Cartesian plane. In the unit circle approach, the sine and cosine function are defined as the x- and y- coordinates of a point on a unit circle, for example, $\sin\theta = b$, $\cos\theta = a$ (Figure 2.2).

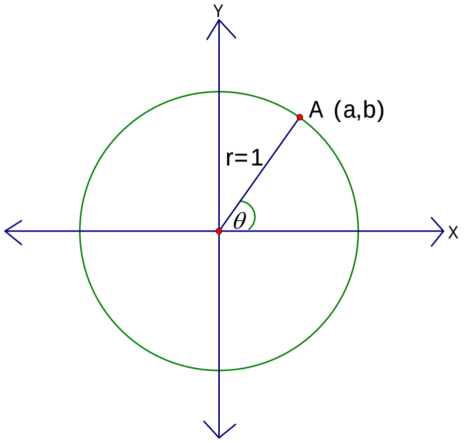


Figure 2.2 Example of the unit circle

2.1.2.1 The components of a circle

A circle is a shape in two dimensions that comprises a point in the middle called the center and a closed curve of equal distance from the center. Figure 2.3 shows the components of a circle, namely, the radius, the diameter, the center, the circumference, the chord, and the arc.

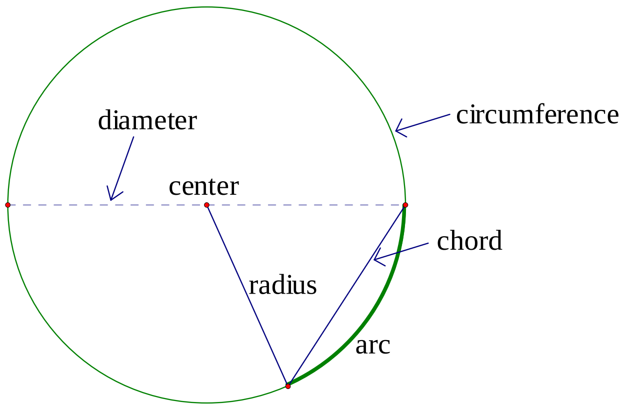


Figure 2.3 The components of a circle

2.1.2.2 π

π is defined by the circumference to diameter ratio. A typical process to estimate π starts by dividing a circle into small, equal sectors (Figure 2.4). An estimate of the arc length of a sector is then the base of the isosceles triangle whose sides are the radii forming that sector. An estimate of π is the sum of these base lengths divided by the diameter. If r denotes the radius, b the base length, and n the number of sectors, we have $\pi \approx \frac{nb}{2r}$. π is irrational that equal approximately to $\frac{22}{7}$.

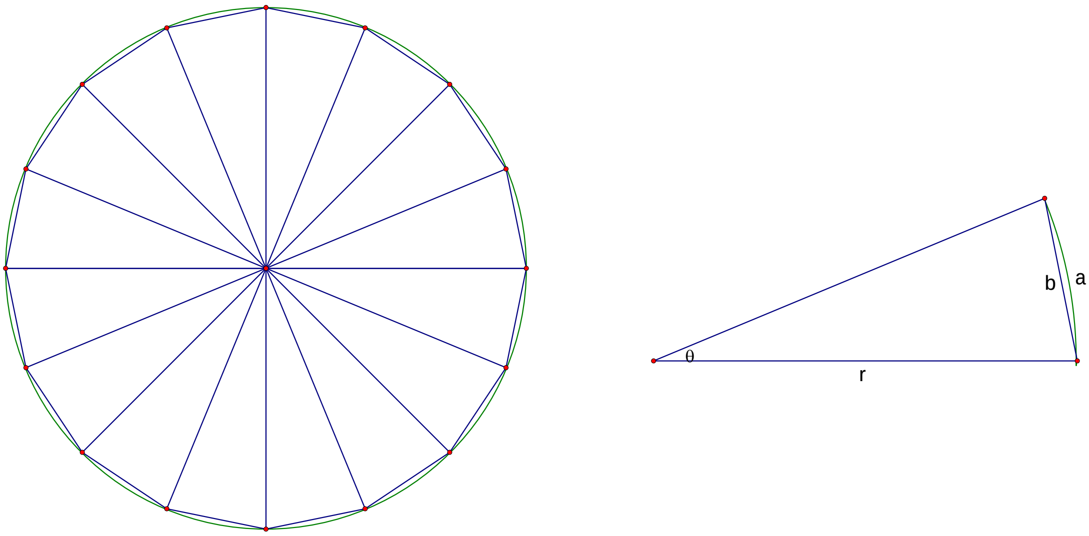


Figure 2.4 A circle with an embedded regular polygon and a corresponding isosceles triangle

2.1.2.3 Radian

Figure 2.5 show that one radian is equal to one radius on the circumference of a circle. There is the arc length of a sector.

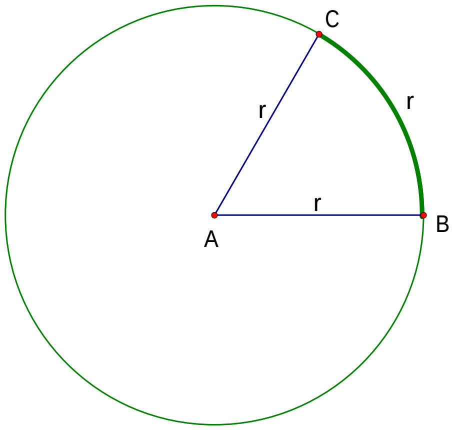


Figure 2.5 One radian on a circle

2.1.2.4 How do you calculate the arc length of a sector?

Let θ denote the angle at the center of the circle of an isosceles triangle (see Figure 2.4), measured in radians. By the definition of radian, the sum of all these angles is 2π .

That is $n\theta = 2\pi$ or $\boxed{\theta = \frac{2\pi}{n}}$.

Let a denote the arc length of a sector. Since the circle is divided into n equal sectors, the circumference of the circle is na. But, by the definition of π, the circumference is $2\pi r$. That is

$$\boxed{na = 2\pi r \Rightarrow a = \frac{2\pi r}{n} = r\left(\frac{2\pi}{n}\right) = r\theta}.$$

In the other way, we determine the arc length of a sector using the relationship between π and the circumference.

By the definition of π, π is the circumference (C) to diameter (2r) ratio. Then, $\pi = \frac{C}{2r} \Rightarrow C = (2\pi)r$. The arc length of a circle is 2π radians. In a full circle, there is 360 degrees (360^0).

We have, 360 degrees are equal to 2π radians.

X degrees are equal to $\left(\frac{2\pi}{360^0}\right) \bullet X^0 = \left(\frac{\pi}{180}\right) \bullet X$ radians.

Thus, the arc length of a circle with a radius of r is $\left(\frac{\pi}{180}\right) \bullet X \bullet r$.

2.2 Inverse Trigonometric Functions and Trigonometric Identities

The basic trigonometric functions are the sine, cosine, and tangent of an angle. Then, the inverse trigonometric functions are as follows:

Table 2.1 The inverse trigonometric functions

Basic trigonometric functions	**Inverse trigonometric functions**
$\cos\theta = a$	$\theta = \cos^{-1} a$ **or** $\theta = \arccos a$
$\sin\theta = b$	$\theta = \sin^{-1} b$ **or** $\theta = \arcsin b$
$\tan\theta = c$	$\theta = \tan^{-1} c$ **or** $\theta = \arctan c$

Many identities interrelate the trigonometric functions. Among the most frequently used is the **Pythagorean identity,** which states that for any angle, the square of the sine plus the square of the cosine is 1. This is easy to see by studying a right triangle of hypotenuse 1 and applying the Pythagorean theorem. In symbolic form, the Pythagorean identity is written $\boxed{\sin^2\theta + \cos^2\theta = 1}$.

Other key relationships are the **sum and difference formulas,** which give the sine and cosine of the sum and difference of two angles in terms of sines and cosines of the angles themselves. One can also produce them algebraically using Euler's formula.

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

2.3 Cartesian Coordinate System

Cartesian coordinate system was used to find the distance between points on two- and three- dimensions are presented. In two dimensions, Cartesian coordinate system is defined as two coordinate from x- and y- axis, to determine the distance between points. For example, let A and B be two-dimensional coordinates on the circle where $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Then, the length of the chord AB is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

In three dimensions, Cartesian coordinate system is also used to determine the distance between points from three axes, x-, y- and z- axis. For example, let A and B be three-dimensional coordinates on the Earth where $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$. We have the length of the straight line AB is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Definition of the Earth Geography

In this research, the developed learning unit on "The Earth and Trigonometry" should not only support students' conceptual understanding of trigonometry in order to solve the shortest distance between points on the spherical Earth surface, but can also integrate mathematical concepts (i.e. trigonometry and geometry) and connect mathematics with the other discipline (i.e. geography) in real-life contexts. Then, this section presents the information of the latitude, longitude, and great circle.

2.4 Earth Geometry

Geography is the science that deals with the study of the Earth and its lands, features, inhabitants, and phenomena. In recent decade, some researchers from Australian Senior Mathematics Journal found the importance between mathematics and geography. In 2004, Faulkner showed how to calculate the distance along a latitude, longitude and great circle using formulae through examples. Quinlan (2006) suggested that "*3D concepts should be introduced via 3D models*" (p. 45). Then, he created the 3D model of the Earth to help the General Mathematics students' understanding the definition of latitude, longitude and great circles (Figure 2.6 and Figure 2.7). The importance of the 3D models is to help students' virtualization in 3D concepts.

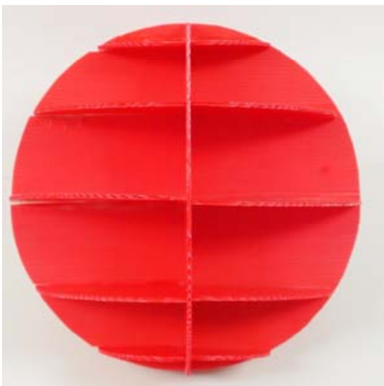


Figure 2.6 A three-dimensional model (Quinlan's) of the earth showing how to measure a latitude

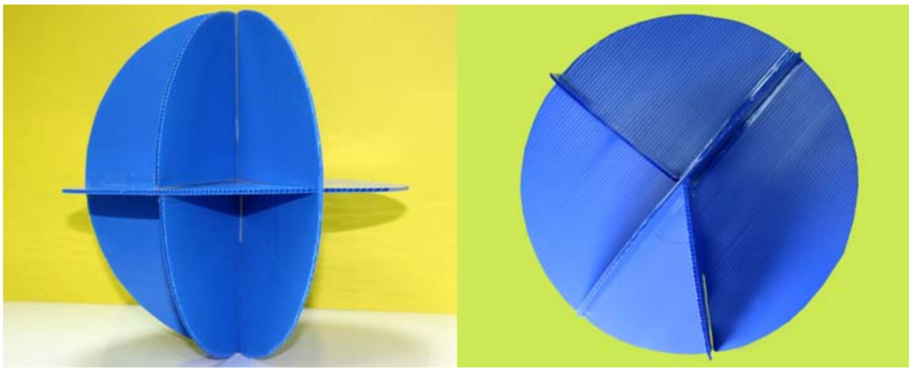


Figure 2.7 The Quinlan's model showing how to measure a longitude: front view and top view

While the shape of the Earth is nearly a sphere, then this study assumes that it is a perfect sphere. The mean radius of the Earth is approximately 6,378 kilometers. A point on the surface of the Earth can also be represented by a Cartesian coordinate system. When referring to such a coordinate system, we will assume that the origin is at the center of the Earth, the x-axis (positive part) goes through latitude 0 and longitude 0, the y- axis goes through the latitude 0 and longitude 90 East and the z-axis (positive part) goes through the North Pole. A specific point on the surface of the Earth is represented by a pair of coordinates. The first coordinate presents the angle of latitude. Another coordinate presents the angle of longitude.

2.4.1 Latitude

The latitude of a point is the angle measured from the plane of the equator to the line joining the center of the Earth and that point. All the points in the same latitude would form a plane parallel to the plane of the equator. Figure 2.9 presents the latitude of a point A, the angle measured is denoted by ϕ .

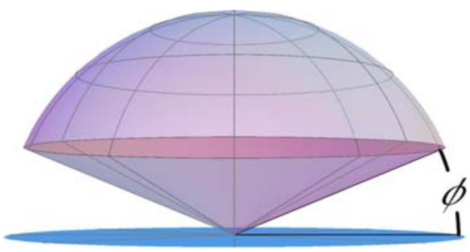


Figure 2.8 A three-dimensional model of the earth showing how to measure a latitude

2.4.1.1 Latitude plane

A characteristic of latitude plane is that of a circle parallel to the plane of the equator. Among entire latitude plane of the Earth is not always equal areas because it is depend on the angle measured of latitude. Note that the biggest latitude plane is the equator.

2.4.1.2 Latitude line

A latitude line is the line around the Earth from right to left or from left to right in the same angled latitude. Because the latitude plane of the Earth is not always equal areas, then the latitude line is not equal circumference of the Earth accepted equator.

2.4.2 Longitude

A longitude of a point is the angle measured from the half circle of the prime meridian to the half circle of the meridian passing through that point. Figure 2.10 presents the longitude of a point A, the angle measured is denoted by λ.

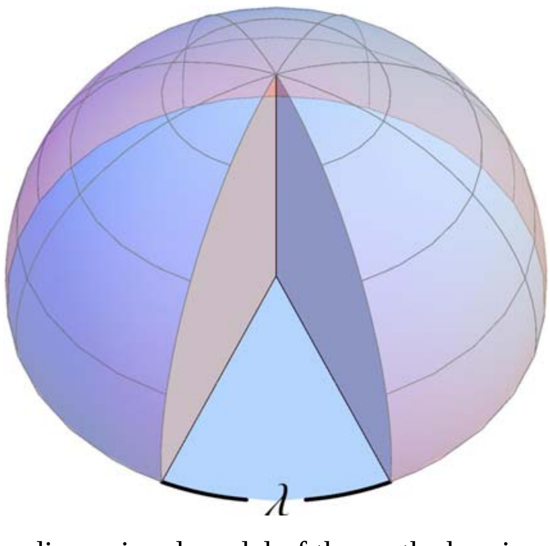


Figure 2.9 A three-dimensional model of the earth showing how to measure a longitude

2.4.2.1 Longitude plane

A characteristic of longitude plane is that of a half circle encompassing the North-South diameter and the half-circle's arc and it is always of equal area.

2.4.2.2 Longitude line

A longitude line is a line goes through the North Pole to the South Pole. Because the longitude plane is always equal areas, then the longitude line is always equal to the circumference.

2.4.3 Great circle distance

A *great circle* is the circumference of a cross section of a sphere that contains a diameter of the sphere. For example the circumference of a cross section along the equator is a great circle whereas the circumference of a cross section along any other circle of latitude is a small circle.

Given any two points on the Earth's surface, there is a unique great circle that contains them. The shortest path between the two points along the Earth's surface is the path along the shortest arc of that great circle. The distance on that path is called the *great circle distance*. Two points that are directly opposite (180^0) to each other along a diameter have to be on the great circle.

Educational Review

Over the years, there are various educational approaches for teaching and learning science and mathematics. Teachers need to consider the effectiveness of the teaching activities in supporting students' understanding. For students, they should be motivated to construct their own understanding of mathematical concepts. The selection of the appropriate approaches in the appropriate circumstances would smooth the way for students to focus on the larger goal of solving the applied problems. This study designed a new mathematical learning unit based on the inquiry approach for high-school students. We posted a problem involving applied mathematics in order to lead students to apply trigonometry to solve real world problems. The problem is to find the shortest path and distance between points (cities) on the spherical Earth surface which is also used to explore students' understanding of Earth geometry and trigonometry.

2.5 Guided Inquiry Approach

Inquiry-based teaching is a pedagogical approach that invites students to explore academic content by posing, investigating, and answering questions. Also known as problem-based teaching or simply as ‘inquiry,’ this approach puts students’ questions at the center of the curriculum, and places just as much value on the component skills of research as it does on knowledge and understanding of content (Center for Inspired Teaching, 2008). Since 1995, Center for Inspired Teaching has helped teachers discover the powerful effects of teaching through inquiry and supported them as they develop the complex instructional skills required to manage an inquiry-based classroom.

The role of the teacher in an inquiry-based classroom is quite different from that of a teacher in a conventional classroom. Instead of providing direct instruction to students, teachers help students generate their own content-related questions and guide the investigation that follows. Because of the role of the teacher in an inquiry-based classroom is unconventional, it is sometimes misunderstood. Administrators, parents, or even students may not recognize the hard work that goes into planning and implementing an inquiry-based approach—in fact, it may seem that teachers “aren’t doing anything” as students struggle to formulate questions and seek out answers. Nothing could be further from the truth. When teachers choose to use an inquiry-based approach, they commit to provide rich experiences that provoke students’ thinking and curiosity; to plan carefully-constructed questioning sequences; to manage multiple student investigations at the same time; to continuously assess the progress of each student as they work toward their solution or final product; and to respond in-the moment to students’ emerging queries and discoveries. The results from Asay and Orgill in 2010 suggest that teachers need opportunities to develop guiding questions that require the use of additional features of inquiry as well as opportunities to facilitate students’ analysis of data, development of evidence-based explanations, and attempts to connect the results of their investigations to accepted scientific principles.

From the perspective of those administrators who have experience introducing and supporting various forms of inquiry into the curriculum, the benefits of inquiry are many (Justice, Rice, Roy, Hudspith & Jenkins, 2009). They believe inquiry can enhance student learning and student performance in other courses. They also believe inquiry can make an important contribution to the way faculty members approach their other teaching responsibilities and that it helps attract and hold onto students. Brown, Wilson and Fitzallen (2007) showed the results from study are extremely supportive of the benefits of introducing inquiry into the mathematics classroom.

Marshall, Horton, Igo and Switzer (2009) provided a definition of inquiry on the survey (*inquiry refers to the development of understanding through investigation, i.e., asking questions, determining appropriate methods, gathering data, thinking critically about relationships between evidence and explanations, and formulating and communicating logical arguments—adapted from the National Science Education Standards, 1996, p. 105*), it is possible that there remain large differences between mathematics and science teachers' understanding not only of what inquiry is, but what it looks like in the classroom. In other words, though the definition for mathematics and science teachers may be the same, science teachers may have a very different worldview than mathematics teachers in how inquiry is best implemented in the classroom.

Just as students are expected to change their role in an inquiry-based classroom, the instructor's role also changes. Instead of simply telling students what to do, instructors using an inquiry- based approach help students find their own answers by asking guiding questions and having the students describe their ideas both verbally and in writing (Gormally, Brickman, Hallar & Armstrong, 2011). Observations and experiences in other classroom has a deep impact on the teacher's understanding of teaching and learning (Carter, 2004). That many other teachers' works have adopted constructivist practices is encouraging. They not only add to existing evidence supporting inquiry-based instruction, they also provide further insights into the complex connections between inquiry and autonomy. In addition to students' developing critical thinking skills by learning through inquiry activities, they also learn to work collaboratively, to articulate their own ideas, and to respect the opinions and expertise of others (Jarrett, 1997). It is important that learning objectives and criteria for success are identified. As a result from research Hanson (2007a) stated that learning is enhanced because the learner feels that the topic is important and worthwhile, the learner has some understanding of what is being learned, and the learner can build understanding from explicit prior knowledge. In 1995, Schiefele and Csikszentmihalyi investigated the relationship between interest, achievement motivation, mathematical ability, quality of experience when doing mathematics, and mathematics achievement. They found that quality of experience with the subject matter was mainly related to interest.

Although inquiry can be described this way, some researchers have found it helpful to demarcate the continuum with descriptive stages—and differentiate whether the instructor or the learner is responsible for developing specified learning outcomes (Hermann & Miranda, 2010). For example,

- in *confirmation,* students are provided with a question, and the results are known in advance;
- in *structured inquiry,* the question and procedure are provided, but students generate an explanation supported by their collected evidence; and
- in *guided inquiry,* the teacher provides the question, but students design the procedures and develop explanations.

Inquiry is a way of learning new skills and knowledge for understanding and creating in the midst of rapid technological change. Inquiry is the foundation of the information age school. The underlying concept is considering a question or problem that prompts extensive investigation on the part of the student. In this sense, it is a research approach to learning. This approach is becoming increasingly common across all subject areas of the curriculum. Inquiry that is guided by an instructional team to enable students to gain a depth of understanding and a personal perspective through a wide range of sources of information is called Guided Inquiry (Kuhlthau, Maniotes & Caspari, 2007, as cited in Kuhlthau, 2010). Guided Inquiry equips students with abilities and competencies to meet the challenges of an uncertain, changing world. Teachers cannot do this alone.

It is well established that the guided inquiry instructional method promotes students' active participation in the learning process. Bilgin (2009) stated that, in guided inquiry method, teachers and learners play a crucial role in asking questions, developing answers and structuring of materials and cases. The usage of guided inquiry method is very important in transition from lecturing method to other teaching methods which are less and more clearly structured for alternative solutions. Guided inquiry activities help students to develop their individual responsibility, cognitive methods, report making, problem solving and understanding skills. Bilgin reported that the greater success and positive attitude toward guided instruction of students in experimental group can be explained as follows: students' participation of teaching materials which is prepared based on the guided inquiry instruction in small groups helped them to acquire meaningful learning by making relationships among concepts and discussion of concepts helped students to recognize their ideas, share their ideas and facilitate their understanding as well as encourage their conceptual restructuring and attitude toward guided instruction.

2.6 Physical Models and Illustrations

2.6.1 Manipulatives

Manipulatives can come in a variety of forms and they are often defined as "physical objects that are used as teaching tools to engage students in the hands-on learning of mathematics" ("Using manipulatives," 2009). Manipulatives can be purchased at a store, brought from home, or teacher and student made. They are used to introduce, practice, or remediate a math concept (Boggan, Harper & Whitmire, 2010).

Swan and Marshall (2010) defined the mathematics manipulative material as an object that can be handled by an individual in a sensory manner during which conscious and unconscious mathematical thinking will be fostered. Consequently, a mathematics manipulative object has the potential to lead to an awareness and development of concepts and ideas linked with mathematics and they would most likely be purpose designed. They concluded that their conclusions were relevant by Perry and Howard (1997). "*Manipulatives benefit the learning and teaching of mathematics. Teacher use of manipulatives needs to be strengthened through appropriate professional development within the overall context of the student's learning of mathematics. There is strong teacher support for manipulative use in the* *earlier grades of primary school. However, all children need access to and availability of a wide range of manipulatives as they meet new mathematical concepts and continue to construct mathematical meanings. ... Schools and education systems need to recognise that the aspirations of their teachers to benefit children's learning to as great an extent as possible mean that manipulatives need to be available to all teachers and all children as they need them. ... This will have budgetary, organisational and professional development implications for the schools and systems*" (pp. 29–30).

Preparing students to use concrete objects in mathematical exploration and problem solving is often overlooked, but is, truly one of the essential elements of successful implementation of a manipulative-based math program (Kelly, 2006). Following these ten essential steps will help to establish a manipulative standard for the classroom:

1. *Clearly Set and Maintain Behavior Standards for Manipulatives*

Students need to have a clearly established criteria for effectively handling and using manipulatives in the classroom. Without a clear set of expectations, students may misuse materials and teachers will become frustrated and disillusioned about manipulative use and, most likely, discontinue their use in the classroom. Rules for specific activities that incorporate manipulatives must be clearly articulated by the teacher, posted in the classroom, and re-affirmed consistently as needed during the manipulative lesson. In short, students need to be meaningfully guided to use and understand the purpose of the manipulative for the specific math task at hand, and then, it will gain relevance for them as mathematicians.

2. *Clearly State and Set the Purpose of the Manipulative Within the Mathematics Lesson*

If students know why the teacher has a certain expectation for a lesson, he/she will be much more likely to attend to the purpose of the task and handle the lesson manipulative correctly. It is important to remember that most math manipulatives are colorful, enticing, and closely resemble what most students have previously referred to as "toys". Since this is a natural association, it is of primary importance that teachers consciously facilitate understanding of the difference between math manipulative or tool and toy. If this is done carefully and effectively at the beginning of the academic year, students will be much less apt to misuse or mishandle mathematics manipulatives.

3. *Facilitate Cooperative and Partner Work to Enhance Mathematics Language Development*

The nature of manipulative use encourages interaction with not only objects but also with people since it usually involves an *action* on an object. Being able to learn and use mathematical language effectively helps to lay a strong foundation for conceptualizing and using abstract math skills in everyday life. It also helps students to develop and feel mathematical power as they become more able to articulate, both verbally and in written form, their math thinking processes. Using partner work with manipulatives to construct mathematical meaning also allows the more reticent math student a supported opportunity to explore strategies from both the observer and participant viewpoints.

4. *Allow Students an Introductory Timeframe for Free Exploration*

Once the purpose and behavior expectations have been established, students need to be given an opportunity to become familiar with the manipulative, discover its properties and limitations, and experiment with it in a variety of contexts. This, too, encourages cooperative work, language development, and risk-taking. Free exploration gives less active students individual opportunities to construct their own meaning and develop confidence in using the manipulative to solidify and enhance their math understanding.

5. *Model Manipulatives Clearly and Often*

Modeling on the overhead or in large or small group sessions will help students see how a particular manipulative can facilitate understanding. For example, when students are beginning to learn about measurement and non-standard or arbitrary units of measure, it is essential that they have a large variety of manipulatives (Cuisenaire rods, Unifix cubes, links, paper clips, pencils, etc.) with which to measure commonly used items (desks, chalk board ledges, window sills, etc.). In doing so, students will be developing real number sense about measurement.

6. *Incorporate a Variety of Ways to Use Each Manipulative*

Offering students different ways to view the same problem will ensure that more of the students will gain a deeper and richer understanding of mathematics. In turn, they will further develop their own levels of fluency and flexibility with numbers. And, showing students how to use the same manipulative in a variety of ways will only tend to strengthen their everyday use and understanding of mathematics. For example, students may use pattern blocks initially to learn colors, shapes, or patterns and eventually create and decipher equal and unequal fractional parts.

7. *Support and Respect Manipulative Use by All Students*

Be sure to clearly and positively "set the stage" in your classroom for inclusion of all students in manipulative use. If you model and use manipulatives, mental models, and other tangible materials to problem-solve, your students will be much more apt to do the same. In turn, when teachers openly express less positive feelings about using manipulatives to solve problems, those who require tangible objects to reach success will be less likely to use them and, subsequently, less likely to gain a firm grasp of the math skill or concept in front of them.

8. *Make Manipulatives Available and Accessible*

In order to facilitate manipulative use at any grade level, the chosen and/or required manipulative must be stored in such a way that it will be physically reachable by all students, plentiful enough (in number) to allow each student to have access to a complete set, and labeled correctly with clear instructions as needed based on the intended purpose, which is not to say that creative manipulative use should not be sanctioned.

9. *Support Risk-Taking and Inventiveness in Both Students and Colleagues*

Teachers who model risk taking and are open to mistakes and re-thinking will enhance student's abilities to move into uncharted territory. Supporting risk-taking and inventiveness in students leads them to explore unknowns and strive to reach unanswered questions as it facilitates open-mindedness and creativity. Students should be supported in seeking and using their own processes in problem-solving. Manipulatives are natural conduits for successful, interactive construction of knowledge through problem-solving.

10. *Establish a Performance-Based Assessment Process*

Since manipulative use is based on constructing or performing an action with a tangible object or set of objects, finding out what students know must also be based on active teacher observation and a set criteria of expected outcomes or, usually, a rubric-style assessment tool. Assessing hands-on inquiry can be a challenging task that implies a commitment of time and energy beyond paper-pencil measures of achievement. Authentically taking into account what a student knows and can do (perform) following a manipulative- and inquiry-based task requires keen observation skills and patience from the teacher. What the teacher actually *sees* the student do with a manipulative-based task is nearly as important as the mathematical thinking that the student can verbally organize and communicate coherently and clearly to teachers, peers, and parents.

Manufacturers advertise manipulatives as materials that will make the teaching and learning of mathematics 'fun' and promote their products as catalysts for engaging students in mathematical learning (Moyer, 2001). Boggan, Harper and Whitmire (2010) reported that elementary teachers who use manipulatives to help teach math can positively affect student learning. Students at all levels and of all abilities can benefit from manipulatives. Durmus and Karakirik (2006) investigated the meaningful educational activities and cognitive tools that might improve students' active involvements in the teaching-learning process and encourage their reflections on the concepts and relations. It is claimed that usage of manipulatives not only increase students' conceptual understanding and problem solving skills but also promotes their positive attitudes towards mathematics since they supposedly provide "concrete experiences" that focus attention and increase motivation. A concrete experience in mathematics context is defined not by its physical or real-world characteristics but rather by how meaningful connections it could make with other mathematical ideas and situations.

2.6.2 Illustrations and visualization

Chiappini and Bottino (1999) considered Mathematics as a domain of knowledge that is concerned with "mathematical objects", that were abstract objects; indeed mathematics objects were not amenable to any concrete imagination or manipulation; they are immaterial, not tangible and accessible. In mathematics learning, differently from the physical concrete world, the learning object cannot be shown in an obstensive way, can be only conjured up by means of the use of external representations. Mathematical concepts such as numbers, functions, vectors, (which are not objects in a usual manner, but which embody relationships) are not directly accessible through everyday experience nor within intuitive perception, as for instance real or physical objects were, but they had to be represented by signs or symbols. This was true also in the case of Euclidean geometrical learning, where the perception involved in managing external representations (drawings) can be also an obstacle for the construction of a mental image. Gaisman and Planell (2007) reported that few students could coordinate the different representations of fundamental planes and surfaces, in some cases because of problems with visualization. They exhibited difficulties with actions that involve intersections between these objects or projections of those intersections in a two dimensional plane. It seemed that when some students were asked to draw on a two-dimensional plane, they interpreted this instruction as drawing on the *xy* plane. It was difficult for them to draw a projection on a plane in 3-D, or even think about it. When asked to draw on a two dimensional plane the projection of the intersection between a fundamental plane and a surface, several students insisted on drawing it in the three dimensional space.

Guven and Kosa (2008) reported that although the students learn three dimensional objects and their features in early stages of elementary school and study three-dimensional objects in various lessons in Turkey, the averages are low. Because the research was carried out with student mathematics teachers, the researchers assumed that students' spatial skills might be high at the beginning of the study. They summarized that the most important of the reasons for the low scores was presenting three-dimensional spatial information in a 2 dimensional format on the blackboard in traditional geometry lessons in Turkey. Because of this limitation students do not have opportunities to create and manipulate 3D models that have vital importance of developing spatial skills. Subramaniam and Padalkar (2009) stated that visuospatial reasoning processes need to be implemented on suitable external representations, which need to be selected and generated in the first place. They have provided illustrations of how diagrams can be powerful tools in visuospatial reasoning. Visualisation is also aided significantly by a suitable choice of familiar situations analogous to the target situation.

According to the results of Yolcu and Kurtulus (2010), primary students can visualized the three dimensional given plane images and in contrast can see the two-dimensional given three-dimensional concrete models, teachers can use different dynamic computer software with plane images, concrete models. Consequently, Students can see plane image on screen and can visualize the three-dimensional.

Guven and Kosa (2008) addressed that the ability of individuals to visualize and manipulate mental images has been recognized as an important cognitive ability in both mundane activities and academic endeavors. Therefore, students' spatial skills should be generally determined and the other reasons of the fail should be examined in Turkey with comprehensive researches.

The following computer programs may be of help to "3-D" representation and simulation on a computer. For example, the *Geometer's Sketchpad* has gained much popularity in mathematics classrooms, Manouchehri, Enderson, and Pugnucc (1998) referred to it as *GSP*. The *GSP* allows the construction of geometric objects and permits the measurement of many attributes. The software is dynamic and user-driven and designed to give students a workspace or worksheet and various tools that construct points, circles, lines, line segments, and rays. Mauricio Ruben and Gonzalo (2010) explored Hilbert geometry in a triangle, using *Maple*, to illustrate some concepts such as Hilbert distance, projective and affine coordinates, unitary circle, etc. and to introduce new "trigonometric functions" for this geometry. Mathematica is a powerful computational tool and an aid to visualization. Rossman, Hawkins, Myers, Melka and Chen (2003) illustrated the use of *Mathematica* in studying space curves in elementary differential geometry. Feedback indicated that students grasped concepts more readily when they were able to visualize the geometry of curves in 3-D by their own efforts. More importantly, they discovered that programming the sequences of formulas encountered required a good understanding of the underlying concepts. We also note that our students introduced to Mathematica in Calculus have continued using it successfully in subsequent courses.

2.7 Theoretical Framework of Education Research

Guided inquiry approach

The guided inquiry instructional method encourages students to actively participate in their learning with the help of the teacher acting as facilitator. As a result, students gain ability to analyze, synthesize, evaluate, and relate what they have learned to other disciplines and everyday life (Jarrett, 1997; Gialamas, Cherif, Keller and Hansen, 2000). Students' conceptual understanding increases in the process of constructing their newly acquired knowledge. When students learn to work cooperatively not only do they acquire social skills but also develop thinking skills (Jarrett, 1997). Hanson (2007a) stated that active participation in learning makes the learner feel the relevance and importance of the topic and hence more interest in learning it. In 1995, Schiefele and Csikszentmihalyi found that interest led to higher motivation, quality of mathematical experience and better mathematical ability and achievement. Gialamas, Cherif, Keller and Hansen (2000) suggested that the instructor might benefit from using other disciplines to reinforce mathematical concepts in their guided inquiry approach. The ventures into other disciplines provide alternative means for assessing students' conceptual understanding. However, students' learning should be assessed by appropriate criteria for success and the learning objectives.

Debriefing at the end of a learning unit in which teacher ensures the overall understanding of concepts should be encouraged (Steinwachs, 1992; Towers & Simmt, 2007). Students and teacher should also discuss what has been missed during the instruction and what should be modified for better learning in the future. For an increasingly effective instruction, teachers should share their experiences from their class with the aim of improving upon their practice; e. g., preparation, actual instructional process, in class review sessions, the so-called lesson study process, especially, for mathematics instruction (Baba, 2007). From these lesson study sessions teachers should be able to generate questions about their practice, design lessons to address problems that have arisen and look for evidence that may help to make their practice better (Fernandez, Cannon, Chokshi, 2003).

Physical models and illustrations

It is generally recognized that in their learning, students make use of their multiple intelligences differently, e. g., some are more abstraction-oriented while others are more comfortable with physical and visual things. For the latter, especially, physical models and illustrations are helpful toward better learning and conceptual understanding. McNeil & Jarvin (2007) saw the above-mentioned manipulatives as an additional resource for learning mathematics and for helping learners to connect with the real world and to increase their memory and understanding. Manipulatives are used to introduce, practice, or remediate a math concept (Boggan, Harper & Whitmire, 2010). Preparing students to use concrete objects in mathematical exploration and problem solving is often overlooked, but is, truly one of the essential elements of successful implementation of a manipulative-based math program (Kelly, 2006). However, there are also alternative opinions about the use of manipulatives such as the gimmick of and the fun derived from these manipulatives may take priority over deep learning and the problem of redundancy in having an extra representation (physical model) for the real things. Perhaps, lesson study sessions (Baba, 2007) by a group of teacher should reduce the highly familiar and/or perceptually interesting manipulatives and emphasize more on the real benefits of classroom practices using manipulatives. In terms of mathematics instruction, manipulatives, rather than the worksheets, may be used creatively to better conceptual understanding, help students connect the concrete with the abstract and match learning styles of some students (Furner, Yahya and Duffy, 2005).

Arcavi (2003) stated that visualization, as both the product and the process of creation, interpretation and reflection upon pictures and images, is gaining increased visibility in mathematics and mathematics education. The study of Kikas (2006) showed that, apart from visuo-spatial ability, verbal ability is also important for children. The role of verbal abilities becomes even more visible in higher grades when more abstract topics are taught. Some types of conceptualization are mediated by signs (verbally or using models, schemas, pictures). Many people struggle with understanding because they have to integrate visually and verbally perceived fragments of the world.

CHAPTER III
METHODOLOGY

Overview

This chapter presents a newly mathematical learning unit based on inquiry approach for high-school students. The development of the learning unit on "The Earth and Trigonometry" is also described. Next, we provide details about the pilot study, the participants and the implementation of learning unit which includes three activities. This chapter also presents the procedures used for data collection and data analysis.

3.1 Methodology

3.1.1 Mixed methods approach

In order to address the research questions, the researcher have opportunities to employ qualitative and quantitative methods, called mixed method approach (Brannen, 2005). This method contributed to the strength of the research strategy (Dunn**,** 1999)**.** Mixed-method approach led to an expansion in both quantitative and qualitative knowledge: 1) quantitative information estimating the size and direction of impacts and 2) qualitative information describing the processes by which impacts occur. In this research, we used mixed method approach to evaluate students' understanding of the learning unit on "The Earth and Trigonometry" and students' perception toward the unit.

3.1.2 Data collection in education research

3.1.2.1 Conceptual understanding test

A conceptual understanding test, as a quantitative tool, was used to assess students' knowledge. A test may be developed and administered by an instructor, a teacher, a clinician, or a test provider. Badger & Thomas (1992) suggested that the use of open-ended questions focused on students' understanding, their ability to reason, and their ability to apply knowledge in less traditional contexts. Such questions can communicate levels of student achievement more clearly than multiple-choice items and give better guidance for instruction. This research study developed the conceptual understanding test, as open-ended questions, to evaluate the students’ understanding of applying the Earth by applying basic trigonometry.

3.1.2.2 Worksheets

A worksheet is a sheet of paper on which problems are worked out or solved and answers recorded. Students in a school may have 'fill-in-the-blank' sheets of questions, diagrams, or maps to help them with their exercises. Students will often use worksheets to review what has been taught in class. To begin to explore students’ knowledge of Earth geometry and trigonometry, a series of open-ended tasks were used. In this case, we used the worksheets as tasks for students. Thompson, Carlson and Silverman (2007) noted that “… *one must design tasks with the learner in mind. An ‘informed’ design is then one in which the designer is guided by a model of the learner*…” (p. 416). Thompson goes on to emphasize the importance of the learner’s understanding of the context and the assigning the task in the model of the learner. In this research the focus was on developing worksheets on collaborative activities for students’ understanding of Earth geometry and trigonometry in their group work. Knowledge of the students and their disposition to explain their thinking in the context of discussions with peers and teacher were critical in securing meaningful evidence of understanding.

3.1.2.3 Students’ reflection

Authentic and meaningful reflection is an important part of the learning process widely adopted by professionals in evaluating their practices. Students reflect on their understanding of what they have learned from activities (Pappas, 2010). Authentic and meaningful reflection has been used at all levels of education in particular to judge the quality of teaching. Reflective practice involves teachers’ examination of the efficacy of their practices.

3.2 Research Design

A scheme of research design and the methodology for this study is shown in Figure 3.1.

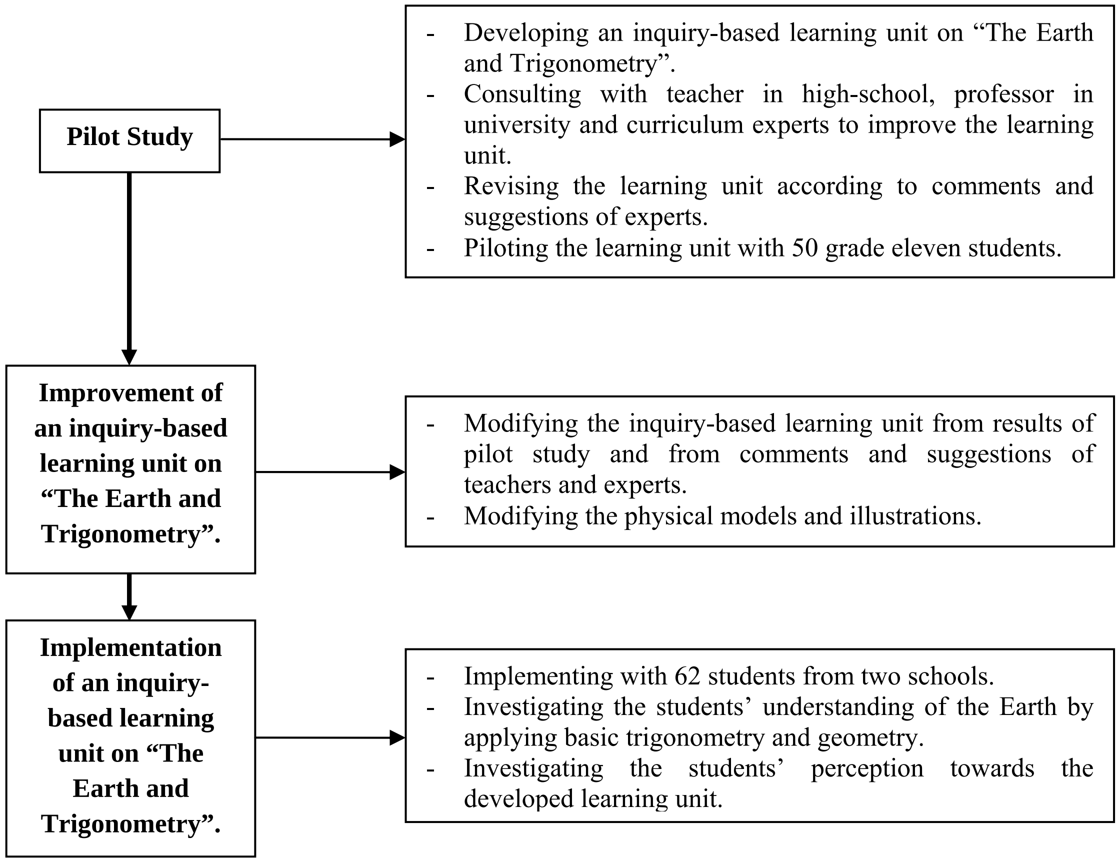


Figure 3.1 Research design for development and implementation of an inquiry-based learning unit

This research design was divided into four main stages described as follows:

3.2.1 Pilot study

The earlier versions of the guided-inquiry learning unit on "The Earth and Trigonometry", which included three physical models and two illustrations, were tried out with 252 secondary students from five schools and modified after each trial (Table 3.1).

Table 3.1 The number of students from five schools

School	Number of students		
	Male	**Female**	**Total**
1	10	25	35
2	15	33	48
3	27	39	66
4	18	35	53
5	18	32	50
Total	88	164	252

We used the lesson learned from our observation and comments and suggestions of students and teachers to gradually modify the learning unit. The major revisions were mainly for the models and illustrations which were meant to help enhance students' learning. The illustration on "converting latitude and longitude to Cartesian coordinates" was made to facilitate students' better understanding of the most difficult topic, the great-circle distance. Moreover, we put more emphasis on encouraging students to ask questions, argue and discuss with peers. The students were guided on how to work cooperatively.

3.2.2 Participants

For the latest pilot study 62 eleventh-grade students from two schools in a big city participated in the study. They were 35 moderately achieving students (10 males and 25 females) of school A and 27 low achieving ones (8 males and 19 females) of school B. The students had already taken trigonometry in the previous semester. All participants, aged 16-18 years, were enrolled in a science program.

3.2.3 Development of the learning unit on "The Earth and Trigonometry"

The learning unit on "The Earth and Trigonometry" was developed based on the guided-inquiry approach with the use of the physical models as learning tools. The learning unit began with a thought experiment involving imagining a sequence of activities that could help students find the solutions to the problems while cooperating with their peers. It was aimed to be used as a supplementary lesson to that in the standard curriculum (IPST, 2001; ACARA, 2010). The content of the learning unit was, nevertheless, based on the standard curriculum. The physical models of the Earth were employed as hand-held learning tools to lead students into the authentic problems of the Earth Geometry.

The lesson plan comprised three main activities to help students construct conceptual knowledge on (i) latitude and longitude, (ii) applied trigonometry, and (iii) the great-circle distance. The two-period (2 h each) learning sequence is illustrated in Table 3.2. Groups of students conducted each activity by themselves while using the given tools along with the guided protocol in the worksheet. Teacher helped in scaffolding the instruction along the way. Students in each group worked cooperatively in collecting, analyzing and interpreting data. They were encouraged to discuss and draw their own conclusions. Students' understanding in each activity was probed by students' answers to questions in the worksheet.

The first activity aimed to help students review the basic trigonometry learned previously, it helped students understand the concepts of latitude and longitude. Two dimensional physical models of the Earth in the form of the Quinlan's model, and the transparent spherical model, and one illustration on measuring the latitude and longitude were used in this activity. The students were expected to know how to measure angles of latitude and longitude and compare the distances between points on the circles of the latitude and longitude.

The second activity was based on the illustrations on measuring the angle of the latitude and longitude. The students then underwent the activity and derived their own knowledge as stated in activity 1. At the end of this activity the students were expected to be able to calculate the arc length between points along the circle of latitude and longitude.

The third activity used the physical models with circular sectors of different arc lengths and the illustration for helping to convert the latitude and longitude into Cartesian coordinates. The students should be able to derive the concepts involved in the great-circle distance as well as find the actual distance along the great circle.

3.2.4 Implementation of the learning unit

The final learning unit as modified from the pilot studies was implemented to two groups of students from two schools according to learning sequence in Table 3.2. During the learning process, the teacher acted as facilitator to encourage students to conduct the assigned activity on their own according to the guided protocol. The teacher closely observed students' participation in their activity and gave suggestions when necessary.

Table 3.2 The learning sequence on "The Earth and Trigonometry"

Phase	Activity	Assessment
Pretest (10 min)	Students undergo pretest	pretest questions
Phase I (15 min) Engagement	- Teacher engages the students with a globe to allow students to observe locations of countries on different continents. - Teacher asks guiding questions, e. g., "What shape does the Earth have?" and "If you want to fly from your country's capital to your destination, what should the flight path of an airliner look like?" Students are encouraged to discuss extensively in small groups. - Each group gives answers and/or poses related questions.	classroom observation
Phase II (60 min) Activity 1 (Basic trigonometry, and latitude and longitude)	- Teacher briefly explains activity 1. - Each group of 4-5 students receives the Quinlan's model and the transparent model of the Earth along with the radian ruler, and one illustration on measuring the latitude and longitude. - Students in each group follow the activity in sequence according to guiding questions in the worksheet. - Group of students collect data, interpret data, discuss and come to a conclusion as a group on the following points, • relationship between angle and arc length	classroom observation group worksheet

Phase	Activity	Assessment
	• definition of the angle; how to measure angles of latitude and longitude • comparison of distances between points along a circle of latitude and longitude (without calculation). - Each group of students presents findings to class. - Teacher conducts whole class debriefing session to ensure students' understanding. - Each group of students gives answers to questions in the worksheet.	
Phase III (60 min) Activity 2 (Applied trigonometry)	- Teacher briefly explains activity 2. - Each group of 4-5 students receives illustrations on angles of latitude and longitude. - Students in each group undergo the activity according to guiding questions in the worksheet. - Students collect data, interpret data, discuss and come to a group conclusion on how to calculate the arc length between points along a circle of latitude and longitude. - Each group of students presents findings to class. - Teacher conducts whole class debriefing session to ensure greater overall understanding. - Each group of students gives answers to questions in the worksheet.	classroom observation group worksheet
Phase IV (60 min) Activity 3 (Great-circle distance)	- Teacher briefly explains activity 3. - Each group of 4-5 students receives the physical model of circular sectors and the illustration on converting latitude and longitude to Cartesian coordinates. - Students in each group do the activity according to guiding questions in the worksheet. - Students collect data, interpret them, discuss and come to a group conclusion on how to calculate the shortest distance (arc length) between points along a great circle. - Each group of students presents findings to class. - Teacher conducts whole class debriefing session to ensure greater and overall understanding. - Each group of students gives answers to questions in the worksheet.	classroom observation group worksheet

Phase	Activity	Assessment
Posttest (20 min)	- Students complete the posttest. - Students write their reflections on understanding and perception on the learning unit	posttest questions students' reflection students' perception

3.2.4.1 Activity I: Duration 60 minutes

1. The learning unit begins with an activity that aims at constructing a coherent concept of angle measure. The researcher asks students to position the transparent model (see Figure 3.2) so that the skewer is vertical with the circular protractor face up, and mark the point two radians to the right of the dot using a string and the radian ruler (see Figure 3.3(a)) whose scale matches that of the transparent model (one radian is equal to the radius of the model) and another point one radian above the dot. The (longitudinal) angle of the first point can be read from the circular protractor and the (latitudinal) angle of the second point can be measured using the properly scaled quarter-circular protractor (see Figure 3.3(b)).

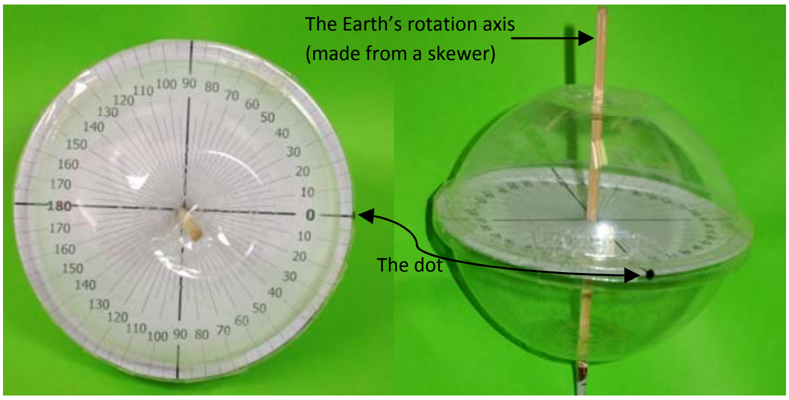
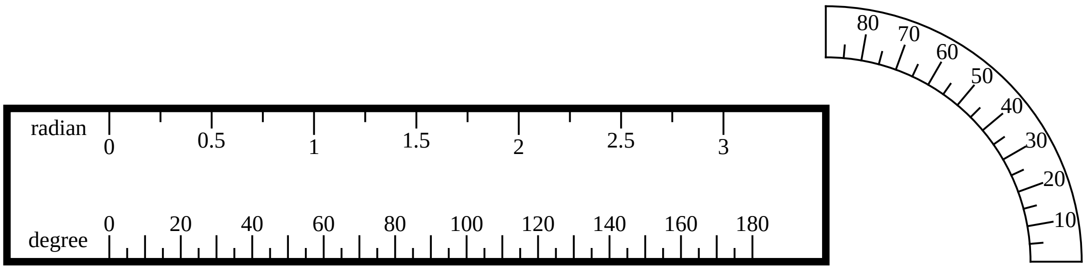


Figure 3.2 A transparent model of the Earth with the dot located at latitude and longitude 0



(a) (b)

Figure 3.3 The radian ruler (a) and the quarter-circular protractor (b)

2. Students were asked to relate the angle obtained from the protractor to the length measured from the radian ruler. The degree side of the ruler should help them realize that an angle can be measured by its subtending arc length. In fact, this is the definition of an angle when one uses radian as the unit of measurement. Measuring an arc length along the surface of the model is easier and more accurate than doing it on a sheet of paper.

3. Students were asked to apply their understanding of angle measure to the construction of the radian ruler for the globe and repeat the measurements using the newly made radian ruler (and the globe). They should see that, with the properly scaled radian ruler, the angles remain unchanged.

4. The researcher let students draw a few latitudinal lines on the transparent model and ask them how to measure the latitudinal angles of these lines. If students seem to struggle, we could remind them of the way they measure angles in the previous activity, or we could recommend using Quinlan's model as a visual aid (see Figure 3.4). If students still cannot do the measurement, we could show them Figure 3.5 (the one without ϕ).

Figure 3.4 Quinlan's (2006) three-dimensional model of the Earth

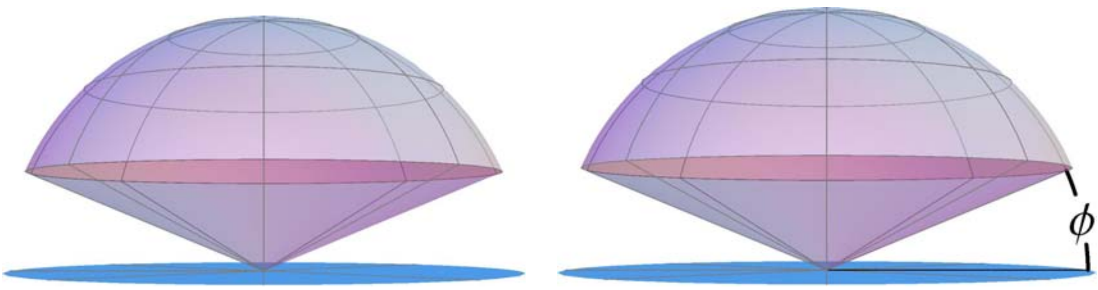


Figure 3.5 Measuring a latitudinal angle

5. The researcher then ask students about the shape of a latitudinal line (circular), the relative sizes of these lines (decreasing from the equator to the Poles), and the characteristics of latitudinal planes (they are parallel to the equatorial plane). We make sure that students' understanding of the latitude encompasses the angle measure in Figure 3.5.

6. The activity for the understanding of longitude is similar to that for the understanding of latitude, with Figure 3.6 replacing Figure 3.5. However, keep in mind that longitudinal angles are measured relative to the Prime Meridian, which passes through Greenwich, England, and longitudinal lines are semicircles whose diameter is the line joining the North and South Poles.

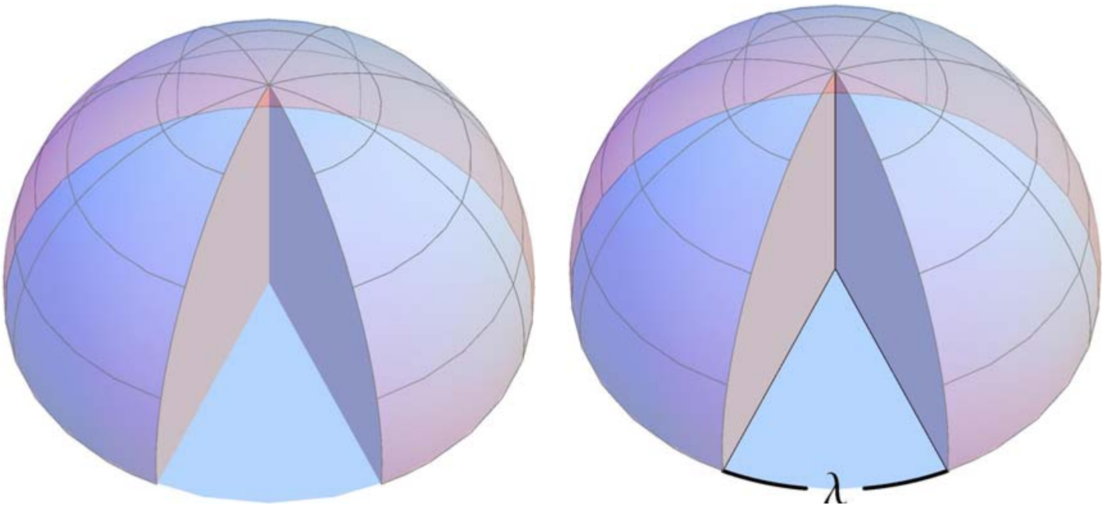


Figure 3.6 Measuring a longitudinal angle

7. To provide students with an opportunity to apply their knowledge of Earth geometry, the researcher specifies the coordinates of a number of points and asks them to locate and mark the points on the transparent model (see Figure 3.2). Some of these points share the same latitude while others share the same longitude.

8. The researcher then ask students to compare the distances between two pairs of points along different lines of longitude sharing the same latitudinal difference (they are equal) and vice versa (the pair closer to the equator is farther apart).

9. The researcher inform them that semicircles of longitude are parts of *great circles*, circles on the Earth surface centred at the centre of the Earth, and opposite semicircles form a great circle, whereas circles of latitude besides the equator are *small circles*.

3.2.4.2 Activity II: Duration 60 minutes

Distance along a circle of longitude

1. Having gone through the first three activities, students realize that they can apply the definition of angle measure to find the distance between two points along a circle of longitude (multiply the (smaller) latitudinal difference, in radians of course, by the radius of the Earth).

2. To find the distance along a circle of longitude, if R denotes the radius of the Earth and φ denotes the latitude, the distance along the circle of longitude would be $R\cdot\varphi$.

Distance along a circle of latitude

3. To find the distance along a circle of latitude, one needs to know the radius of that small circle, which can be derived from Figure 3.5. If R denotes the radius of the Earth and φ denotes the latitude, the radius of the circle of latitude would be $R\cdot\cos\varphi$. Figure 3.4 or Quinlan's model should help students see that the longitudinal differences are the same across all circles of latitude. The problem is then reduced to just another application of the definition of angle measure.

4. However, students may not realize that travelling along a circle of latitude is not optimal because it is not a great circle. The next activity is designed to help them on this issue.

3.2.4.3 Activity III: Duration 60 minutes

The great-circle path

1. Students were asked to mark on the transparent model two points that are on latitude 60° N and on two opposite longitudes, for example, 60° N 30° E and 60° N 150° W, and calculate the distances along the circles of latitude and longitude ($\pi R/2$ and $\pi R/3$). Since a circle of longitude is a great circle, the latter distance is the shortest between the two points along the Earth surface.

2. To convince students that this is the case, the researcher utilize circular sectors whose radii are the same and whose central angles are 60°, 90°, and 180°. The cone made by joining the two radii of the last sector can be perfectly bisected by inserting the pointed end of the first sector into it (see Figure 3.7). The arc of the bisecting sector represents the path along the circle of longitude while that of the ninety-degree sector inserted into the same cone represents the path along the circle of latitude.

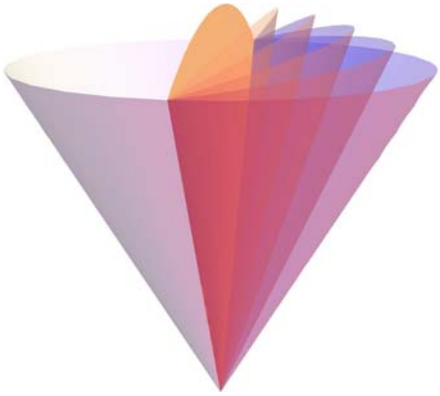


Figure 3.7 Circular sectors inserted into the same side of the cone

3. Fixing the bisecting sector in the middle and inserting common-radius sectors whose central angles are 62°, 66°, 72°, and 80° into the same side of the cone gives us the cone in Figure 3.7. (A better alternative would be to use a more durable plastic cone and adjust the circular sectors accordingly.) They should see that the surface of the bisecting sector is flat while those of the others are curved, indicating that the great-circle path is the shortest one, comparable to a straight line being the shortest path between two points.

4. Students should also notice that the arcs of the sectors outline a sphere's surface, indicating that these arcs represent paths on the surface. Since the arc lengths can be directly compared and the bisecting sector has the shortest arc, the same conclusion can be reached more concretely.

5. The arcs of the circular sectors other than the sixty-degree one are parts of the small circles, some of which are depicted by the white circles in Figure 3.7. The circumferences of these small circles are the same as those of the circles at latitudes 37° 34' 16", 50° 38' 8", 56° 40' 47", and 59° 18' 7". So if the researcher tilt the cone until one of the arcs is horizontal, the arc of the sixty-degree sector would represent the great-circle path (and distance) between the two points on the corresponding latitude (see Figure 3.8).

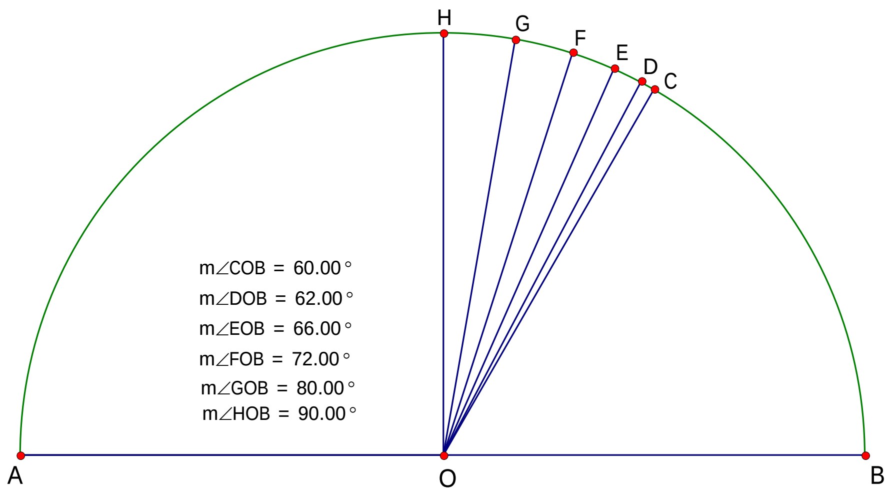


Figure 3.8 Circular sectors tilted whose central angles are 62°, 66°, 72°, and 80°

6. Afterwards, students can mentally construct the circular sector whose arc represents the great-circle path between any two points. Thus the problem of finding the great-circle distance between points *A* and *B* is reduced to finding the central angle, θ, of the sector in Figure 3.9.

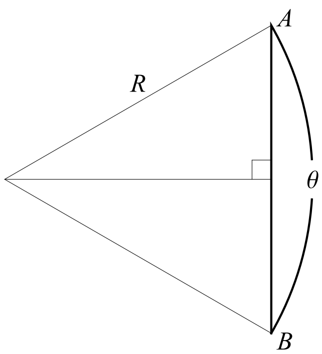


Figure 3.9 The circular sector whose arc represents the great-circle path between *A* and *B*

The Great-circle Distance

7. Given the distance $\overline{AB}$ (through the Earth), simple trigonometry tells us that $\sin\frac{\theta}{2} = \frac{\overline{AB}}{2R}$, which implies that $\theta = 2\arcsin\frac{\overline{AB}}{2R}$.

8. And since the required distance can be found from the Cartesian coordinates of *A* and *B*, we should turn students' attention to the task of converting latitude-longitude coordinates into Cartesian ones after informing them that the origin of the Cartesian coordinate system is at the centre of the Earth, the positive part of the *x*-axis goes through latitude 0 and longitude 0, and the positive part of the *z*-axis goes through the North Pole.

9. For the groups that cannot accomplish the task, the teacher tells them to first consider only the plane of latitude of point *A* (or *B*) and try to find its *x*- and *y*-coordinates with the help of Figure 3.6. They see that the longitudinal angle is the same across all parallels of latitude and thus the coordinates depend on the radius of the circle of latitude of that point, which is already known from a previous activity.

10. The *z*-coordinate can be obtained from the height of the parallel of latitude, which can be derived from Figure 3.5 or from Quinlan's model (see Figure 3.4). If they still have difficulties, we should show them Figure 3.10.

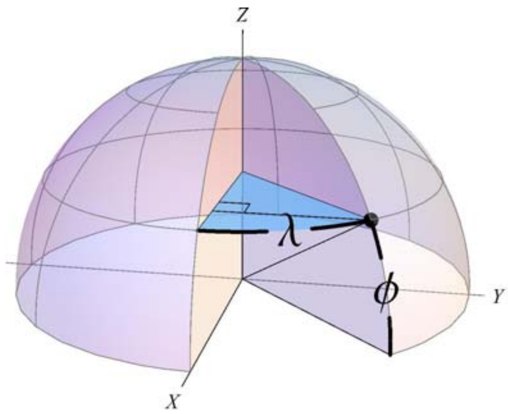


Figure 3.10 Converting latitude and longitude to Cartesian coordinates

11. In conclusion, the Cartesian coordinates of the point at latitude φ and longitude λ are $(R\cdot\cos\varphi\cdot\cos\lambda, R\cdot\cos\varphi\cdot\sin\lambda, R\cdot\sin\varphi)$.

12. Suppose students find that the Cartesian coordinates of the two points are (x_A, y_A, z_A) and (x_B, y_B, z_B) and they seem to struggle with finding the distance between the points, we can suggest considering the horizontal distance first

(between, say, (x_A, y_A, z_A) and (x_B, y_B, z_A)) and ask whether they see the right triangle whose hypotenuse is the required distance.

13. Given the chord length and still students have trouble finding the central angle (see Figure 3.9), we should draw the figure for them. Make sure that they measure the angle in radians and multiply it by the radius of the Earth to get the great-circle distance.

3.3 Data Collection and Data Analysis

The data collected included the conceptual understanding test, the worksheet, students' reflection on their understanding and students' perception on the learning unit. The data on conceptual test were collected before and after the implementation of the learning unit. Students' worksheets were collected at the end of each activity. The analyzed data led to the assessment of the students' understanding of the topics on the Earth and the relevance of trigonometry.

3.3.1 Conceptual test

The conceptual understanding test, as a quantitative tool, was used to assess the students' understanding on the Earth and trigonometry. In this research, the pre- and post-conceptual tests were designed to have six parallel items. Each item allowed students to fill in their answers and explain their thoughts. At the beginning, the pre-conceptual test scores were used to assess students' prior knowledge whereas the post-conceptual test scores were used to follow students' achievement after the intervention.

The conceptual understanding test was divided into four topics as follows:

1. *Basic trigonometry* was given to investigate students' prior knowledge in trigonometry including the Pythagorean theorem. It contained five questions; each question required them to know the trigonometric functions such as sine, cosine and tangent and how to calculate the length of a missing side of a right angle triangle and a unit circle.

2. *Latitude and longitude* contained two questions that required comparing the arc length between points along the circles of the latitude and longitude.

3. *Applied trigonometry* contained three questions asking students to calculate the length of the missing side of a right triangle embedded in a circle. Another question was used to explore the concept of the radian. The rest of questions required students to calculate the arc length of a sector given the angles at the center.

4. *Great circle distance* contained three questions. Students were given two points on the Earth in terms of latitude and longitude. Students were required to find the Cartesian coordinates for these points, the chord and the arc length of a sector (great circle) respectively.

The questions were validated by three experts, one in mathematics and two in mathematical education. For reliability, the Cronbach alpha was 0.8 for the test on the 150 high-school students. For the constructed-response items, the researcher developed a specific rubric for allocating 1 to 3 points for answers to each part of a question in accordance with criteria for correctness. The total score was 40. The data were analyzed by using SPSS program. The means and the standard deviations were employed for the pre- and post-conceptual tests of students' achievement on individual topics. The paired samples t test at the significance level of 0.001 ($p < 0.001$) was used for comparison of the mean scores between the pre- and post-conceptual test of students' achievement on the learning unit. Due to differences in the prior knowledge between students in the two schools, the results from the two schools were analyzed separately.

The paired samples t test is: $t = \frac{\bar{x}_d}{se_{\bar{x}_d}}$

where $\bar{x}_d$ represents the mean difference in the sample and $se_{\bar{x}_d} = \frac{s_d}{\sqrt{n}}$.

3.3.2 Students' reflection on their understanding

The students were asked to reflect on what they had learned from the learning unit and their understanding of the four topics. The answers from students' writing were categorized and reported as percentage of students' covering the particular type of reflection.

3.3.3 Students' perception of the learning unit

The perception of students on the learning unit was assessed by using open-ended questions. The students were asked to write freely about their thoughts on the following points: comparison between the previous learning activity and the newly developed learning unit, advantages and disadvantages of the learning unit, comments and suggestions. Students' answer were categorized and shown as percentage of students mentioning each particular perception.

3.3.4 Worksheets

The designed worksheets contained the four topics as shown above. Knowledge about the students and their dispositions were derived from their thinking in the discussion with peers and teacher. For the constructed-response items, the researcher developed a specific rubric for allocating 1 to 3 points for answers to each part of a question in accordance with criteria for correctness. The total score was 40. The data were analyzed by using SPSS program. Evidence from each worksheet (the means and the standard deviations) reflected each group of students' understanding on the four topics about the Earth and trigonometry.

CHAPTER IV
RESULTS

Overview

This chapter reports the results of the development and implementation of the learning unit on the Earth and trigonometry. The findings are presented in two main parts. The first part provides the preliminary finding from the pilot study. The other part presents the finding after using the developed learning unit that aimed to promote students' conceptual understanding of the Earth and trigonometry. Students' reflection on their understanding and students' perception of the learning unit are described.

4.1 The Development of the Physical Models and the Illustrations

The development of our instruction began with a thought experiment involving imagining a sequence of activities that could help students find the solutions to the problems in the activities in cooperation with their peers. The instruction was then piloted on prospective students in order to find out whether students could solve the problems and whether they were enthusiastic about the activities. The thought experiment-pilot study cycle was repeated until a proper sequence of activities was obtained. Along the way, we were surprised to find that even though most students were familiar with latitudinal and longitudinal lines and the *names* of these lines (e.g. latitude 32 degrees south, longitude 128 degrees east), they usually did not realize that the word *degree* refers to the angle measure, and did not know how to measure it even among the ones with the realization. We also found that, in general, group activities were more effective than class discussions, specific problems were more suitable than generalized problems, and verbal suggestions needed to be accompanied by visual representations.

4.1.1. The physical models

Three types of physical models were employed to help students visualize Earth geometry.

I. The Quinlan's (2006) three-dimensional model (Figure 4.1) of the Earth was used to help students visualize the latitude and the longitude.

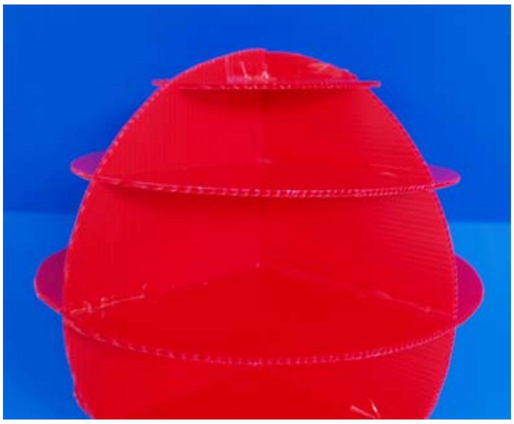


Figure 4.1 Quinlan's (2006) three-dimensional model of the Earth

II. The transparent spherical model (Figure 4.2), the second type, made from readily available hemispherical coffee-cup lids together with angle measuring scales, helped students build up their understanding of the coordinates and the great circle distance. This was modified from a model used in astronomical education (Ruangsuwan & Arayathanitkul, 2009).

Figure 4.2 Transparent model of the Earth

III. In the third type, circular sectors (Figure 4.3) were used to indicate that the great-circle path was the shortest one, an analogy is the straight line being the shortest path between two points on a flat surface of Euclidean geometry. These models can be easily made by teacher and/or students using paper and plastic sheets.

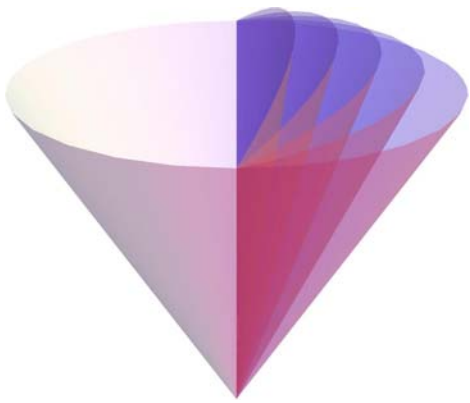
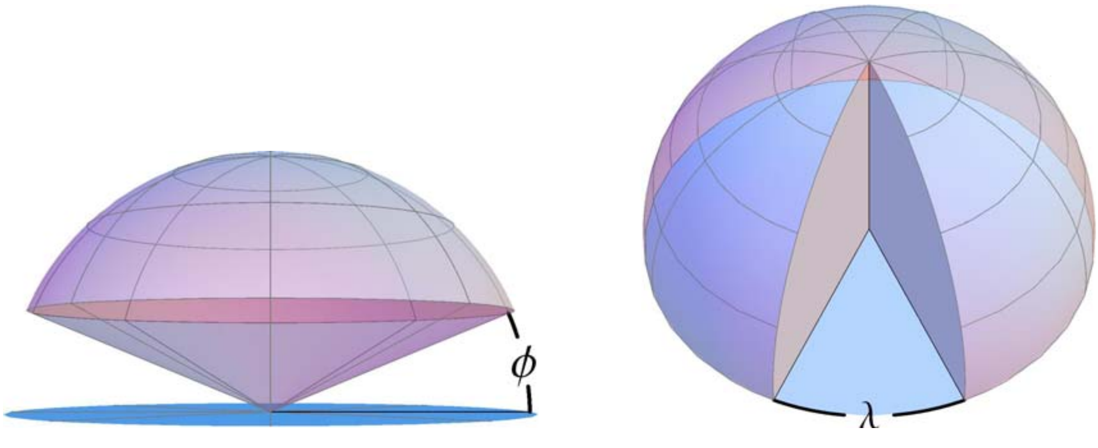


Figure 4.3 Model of circular sectors inserted into the same side of the cone

4.1.2. The illustrations

In addition to the models, two illustrations were offered. The first was a drawing showing how to measure the latitude and longitude angles and to help students calculate the arc length along a circle of latitude and longitude (Figure 4.4).



(a) (b)

Figure 4.4 Illustrations showing measurement of (a) a latitudinal angle (b) a longitudinal angle

The second, is a drawing on the way to convert latitude and longitude to the Cartesian coordinates by helping students transform a point on the Earth surface into its Cartesian coordinates, which were then used to find the great circle distance (Figure 4.5).

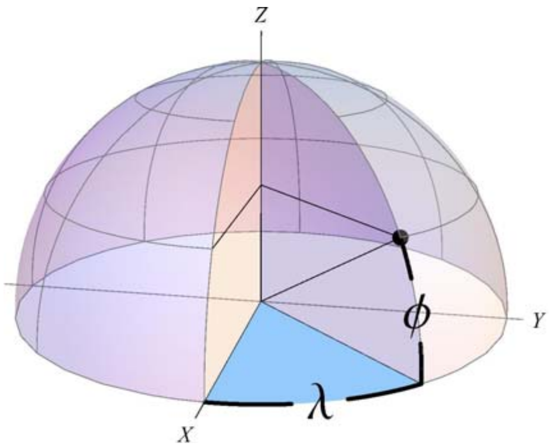


Figure 4.5 Illustrations showing converting latitude and longitude to Cartesian coordinates

4.2 The Effectiveness of the Learning Unit in the Pilot Study

The pilot study mentioned here is the latest one. Results from the ones preceding it led to successive modifications and ending up with the present pilot learning unit. The results in Table 4.1 show the mean scores of students' group worksheets after completing the pilot learning unit which consisted of three main activities. Activity 1 covered the topics on basic trigonometry and latitude and longitude, while activities 2 and 3 focused on applied trigonometry and the great circle distance.

Results from the pilot study showed that after participating in the learning activity the students had good understanding of the first topic on basic trigonometry, and fair understanding in the second and third topics. However, they still had very poor understanding of the last topic (mean score = 2.40 out of 10).

Table 4.1 Mean scores of students' group worksheets from the pilot study (n=50)

Topic	**Pilot study (Mean ± S. D.)**
1. Basic trigonometry (10 points)	8.40 ± 0.89
2. Latitude and longitude (10 points)	6.80 ± 1.10
3. Applied trigonometry (10 points)	6.40 ± 1.67
4. Great circle distance (10 points)	2.40 ± 0.89
Total (40 points)	24.00 ± 2.83

In this pilot study, the scores of worksheets of student groups (Table 4.1) were similar to those of the posttest (Table 4.2) in which students did the test individually. The percentage increases (% posttest - % pretest) were 3.4, 51.6, 40.5 and 11.1 while the posttest scores were 8.04, 5.24, 5.38 and 1.11 out of 10 in topics 1, 2, 3 and 4 respectively.

Table 4.2 Mean scores of pretest and posttest of students from the pilot study (n=50)

Topic	**Pilot study (Mean ± S. D.)**		
	Pretest	**Posttest**	**%posttest-%pretest**
1. Basic trigonometry (10 points)	7.70±1.87	8.04±1.84	3.40
	t = 1.17		
2. Latitude and longitude (10 points)	0.08±0.40	5.24±3.93	51.60
	t = 9.44***		
3. Applied trigonometry (10 points)	1.33±1.41	5.38±2.65	40.50
	t = 12.02***		
4. Great circle distance (10 points)	0.00±0.00	1.11±1.50	11.10
	t = 5.24***		
Total (40 points)	9.11±2.68	19.78±7.08	26.68
	t = 11.73***		

*** significant difference at 0.001 level

These students were among the moderate achievers and it was expected that they would have better achievement if the learning unit was improved. The results thus suggested for a greater effort at improving the models and all the activities, especially, that of activity 3 on the great circle distance. In fact, teacher's observation and students' comments and suggestions were discussed and analyzed to improve the learning unit for the present study.

4.3 The Effectiveness of the Developed Learning Unit

The newly developed learning unit, after modification from the latest pilot study, was implemented to two schools: school A with moderately achieving students and school B with low achieving students.

Table 4.3 Mean scores of pretest of students from schools A (n=35) and B (n=27)

Topic	**School A (Mean ± S. D.)**	**School B (Mean ± S. D.)**
1. Basic trigonometry (10 points)	8.71±1.58	2.44±1.19
2. Latitude and longitude (10 points)	0.46±1.54	0.00±0.00
3. Applied trigonometry (10 points)	1.64±1.99	0.00±0.00
4. Great circle distance (10 points)	0.00±0.00	0.00±0.00
Total (40 points)	12.38 ±4.47	2.44±1.19

Table 4.3 shows the mean scores of pretest of students in both schools that the pretest scores of the students from school A were higher than those in school B in first three topics. However, the students from both schools got zero for the fourth topic.

Table 4.4 Mean scores of students' group worksheets from school A (n=35) and school B (n=27)

Topic	**School A (Mean ± S. D.)**	**School B (Mean ± S. D.)**
1. Basic trigonometry (10 points)	9.82 ± 0.60	9.50 ± 0.93
2. Latitude and longitude (10 points)	9.64 ± 0.81	9.75 ± 0.71
3. Applied trigonometry (10 points)	9.82 ± 0.60	9.50 ± 0.93
4. Great circle distance (10 points)	9.64 ± 0.81	3.00 ± 1.07
Total (40 points)	38.91 ± 2.43	31.75 ± 1.67

Results in table 4.4 show the mean scores of students' group worksheets. There was no significant difference in the mean scores on the first three topics between the two schools. However, in the last topic on the great circle distance, student groups of school B showed a very low mean score (3.00 ± 1.07) compared to a higher mean score (9.64 ± 0.81) in school A. This result, however, was of the group and might not reflect individual understanding.

The results in Table 4.5 show the mean scores of the pretest and posttest on the four topics on trigonometry from students in school A and school B. The pretest scores of the students from school B were significantly lower than those of school A in all four topics tested. Moreover, the students from school B got zero points on the three topics, i.e., latitude and longitude, application of trigonometry and great circle distance. Students from school A also got zero for the fourth topic.

Table 4.5 Mean scores of pretest and posttest of students from schools A (n=35) and B (n=27)

<table>
<tr><th rowspan="2">Topic</th><th colspan="3">School A (Mean ± S. D.)</th><th colspan="3">School B (Mean ± S. D.)</th></tr>
<tr><th>Pretest</th><th>Posttest</th><th>%posttest - %pretest</th><th>Pretest</th><th>Posttest</th><th>%posttest - %pretest</th></tr>
<tr><td rowspan="2">1.Basic trigonometry (10 points)</td><td>8.71±1.58</td><td>9.19±1.56</td><td rowspan="2">4.80</td><td>2.44±1.19</td><td>8.72±1.27</td><td rowspan="2">62.80</td></tr>
<tr><td colspan="2">t = 1.38</td><td colspan="2">t = 20.00***</td></tr>
<tr><td rowspan="2">2.Latitude and longitude (10 points)</td><td>0.46±1.54</td><td>7.20±2.39</td><td rowspan="2">67.40</td><td>0.00±0.00</td><td>6.07±2.04</td><td rowspan="2">60.70</td></tr>
<tr><td colspan="2">t = 13.88***</td><td colspan="2">t = 15.49***</td></tr>
<tr><td rowspan="2">3.Applied trigonometry (10 points)</td><td>1.64±1.99</td><td>7.64±2.99</td><td rowspan="2">60.00</td><td>0.00±0.00</td><td>6.96±1.48</td><td rowspan="2">69.60</td></tr>
<tr><td colspan="2">t = 10.91***</td><td colspan="2">t = 24.44***</td></tr>
<tr><td rowspan="2">4.Great circle distance (10 points)</td><td>0.00±0.00</td><td>7.14±3.20</td><td rowspan="2">71.40</td><td>0.00±0.00</td><td>3.11±2.82</td><td rowspan="2">31.10</td></tr>
<tr><td colspan="2">t = 13.19***</td><td colspan="2">t = 5.73***</td></tr>
<tr><td rowspan="2">Total (40 points)</td><td>12.38 ±4.47</td><td>31.17 ±8.16</td><td rowspan="2">46.98</td><td>2.44 ±1.19</td><td>24.87 ±4.43</td><td rowspan="2">56.08</td></tr>
<tr><td colspan="2">t = 12.66***</td><td colspan="2">t = 26.05***</td></tr>
</table>

*** significant difference at 0.001 level

After participating in the learning unit, the total mean scores of posttest of the students from both schools A and B were significantly higher than that of the pretest, with the percentage increase (% posttest - % pretest) of 46.98 and 56.08 respectively. Regarding the mean score for each topic, there was no significant difference between pretest and posttest in the topics on basic trigonometry in school A, while the posttest scores of the topics 2 and 3 were significantly higher than those of the pretest, with the percentage increase of 67.4 and 60.0% respectively. In the fourth topic, the mean score increased from 0% to 71.4%. In school B, due to very low pretest scores, there was a large increase in the percentage in all topics, i. e., 62.8% in basic trigonometry. In the other three topics, the posttest scores increased from 0 to 60.7, 69.6 and 31.1 respectively. When the posttest score on each topic was compared between the two schools, although school B's posttest scores in the first three topics appeared slightly lower than those of school A this was not statistically significant. However, the mean scores of the fourth topic of school B was only 3.11 (out of 10) when compared to 7.14 of school A. The results showed that students in school B, with very poor prior knowledge on basic trigonometry, albeit much improved, were not be able to learn much from the activity on the great circle distance. Nevertheless, after finishing all the activities (including posttest) the teacher helped in the debriefing again (extra class time) to ensure greater overall students' understanding.

4.3.1 Students' reflection on their understanding

The students were asked to reflect on their understanding of the topic "The Earth and Trigonometry" by writing about what they had learned and level of their understanding. The points raised from students of both schools were similar as shown in Table 4.6. The students' answers were categorized as percentages of students who covered the relevant and important points.

Table 4.6 Students' reflection on their understanding of the learning unit

Topic	**Students' reflection**	**Percentage of students covering the type of reflection (%)**	
		School A (n=35)	**School B (n=27)**
What students learned from the newly developed learning unit	- understood better from models and illustrations	85.7	81.5
	- knew more about geography (latitude, longitude)	77.1	74.1
	- understood the globe better in with a new realization in applying trigonometry, which would make this course more interesting	74.3	74.1
	- knew how to apply trigonometry to other subjects	62.9	77.8
	- learned to work cooperatively with groupmates	28.6	37.0
	- gained other knowledge than trigonometry	20.0	18.5

	Students' reflection	**Percentage of students covering the type of reflection (%)**	
		School A (n=35)	**School B (n=27)**
	- helped to review prior knowledge on trigonometry	11.4	29.6
Students' understanding of the four topics learned	- did not clearly understand some topics, mostly about the great circle distance	54.3	48.1
	- partially understood some topics	42.9	37.0
	- clearly understood all topics	11.4	3.7

More than 50% of the students in school A and school B mentioned about the new knowledge gained about geography.

"I knew more about latitude and longitude"

"I knew more about geography"

"The geographical coordinates are more easy to find now"

About 70% of the students in school A and school B said that *"the new learning unit was easy to understand, compared to the past learning activity which was without any reference to the Earth"*. Moreover, the students commented that *"the activity encouraged us to grasp more concepts"*.

The majority of students (62.9% in school A and 77.8% in school B) stated that the unit helped them to apply trigonometry, for example

"The instruction helped me to apply trigonometry to geography"

"The knowledge gained can be applied to everyday situations"

The students also mentioned that *"they gained new knowledge that they never knew before"*. The students from both schools agreed that it was the models and illustrations in the learning unit that helped them apply trigonometry concepts better. Some students (11.4% in school A and 29.6% in school B) mentioned the first learning activity helped them review their prior knowledge on basic trigonometry and thus facilitated learning and understanding in the topics that followed.

Regarding the level of student understanding, only 11.4% of the students in school A and 3.7% of students in school B showed good understanding of all the topics in the unit. About 40% of the students understood some topics and about 50% of the students voiced that they did not quite understand a particular topic, namely, “great circle distance”. This topic required mastering of the three dimensions for better understanding of the subject. However, 85.7% of the students in school A and 81.5% in school B agreed that the physical models as visual aids had proved beneficial to them as follows:

“The physical models helped me to build up my visualization in 3-D”

“Using the physical models supported my understanding of the 3-D”

“I like to do the activities using physical models”

Figure 4.6 Students’ activity using transparent spherical model

4.3.2. Students’ perception of the learning unit

A questionnaire containing open-ended questions was used to assess students’ perception on the format of the learning unit. The students compared the newly developed learning unit to their past learning about mathematics. The students’ view points were categorized and reported as percentage of students covering the type of reflection as shown in Table 4.7.

Table 4.7 Students' perception of the learning unit

Topic	Students' reflection	Percentage of students covering the type of reflection (%)	
		School A (n=35)	School B (n=27)
The newly developed learning unit activity vs. previous learning activity	- the new learning unit allowed us to ask questions and discuss in class	91.4	88.9
	- we enjoyed learning trigonometry in the new learning unit very much when compared to learning in the past	85.7	88.9
	- past learning was of traditional type with no group activity and very few learning activities	74.3	74.1
	- the new learning unit had interesting models and illustrations to enable better understanding	48.6	55.6
	- group activity in the new learning unit helped us to learn and lessen the work needed for understanding	37.1	48.1
	- more interaction between teacher and students in the new learning unit	34.3	37.0
	- the topic on great circle distance was too difficult to understand	14.3	37.0
	- traditional learning in the past was easier to follow	11.4	11.1
	- the new learning unit was more difficult than the traditional one	5.7	7.4
Advantages of the learning unit	- easy to understand	54.3	48.1
	- enjoyable	42.9	59.3

Topic	Students' reflection	Percentage of students covering the type of reflection (%)	
		School A (n=35)	School B (n=27)
	- more knowledge gained	40.0	51.9
	- opportunity to work cooperatively	31.4	37.0
	- helped us to be more creative and imaginative	2.9	7.4
Disadvantages of the learning unit	- difficult to understand topics that involve 3-D due to lack of imagination	28.6	37.0
	- time allowed for each activity was too short	14.3	14.8
	- not suitable for students who could not combine geography with trigonometry	-	7.4
Comments and suggestions	- teacher should show animated pictures of the models as well	14.3	37.0
	- more exercises should be added in the worksheet	5.7	18.5
	- more time should be given	5.7	11.1

The majority of students (74%) commented that in the past the learning unit on mathematics was of traditional type with no group learning. About 90% of the students were in favor of the new learning unit, although about 10% of them felt that the traditional learning was easier to follow.

"The instruction was very good because it applied the lesson to real-life situations"

"Very good instructional media"

"Appropriate number of activities"

Additionally, about 35% of the students in both schools stated that the format of learning unit allowed more student-student interaction and student-teacher interaction as well as encouraged them to ask questions and discuss in class, for example,

"Group discussion promoted my thinking skill and problem-solving skill"

"I liked to discuss my ideas with others concerning the learning unit"
"I had the opportunity to do the activities with others"
"I had a chance to discuss with friends and the teacher"
"It was a new learning unit that allowed me to have a good relationship with others in the group"

Figure 4.7 Students' cooperative learning with their groupmate

Other comments are
"Enjoyable"
"The learning unit was very good because it applied the lesson to real-life situations"

However, about 14% and 37% of the students in schools A and B voiced that the learning unit was difficult to understand because it involved knowledge about three-dimensions which required a lot of imagination. Most of these were students in school B who seemed to have poor prior background in trigonometry. A few students felt that the learning unit was not suitable for those who could not see the relationship as repeatedly pointed out by the teacher and that these two topics should be taught separately in separate geography and trigonometry classes.

CHAPTER V
DISCUSSION

Overview

This chapter aims to evaluate the findings of the learning unit from the conceptual understanding test, the worksheet, students' reflection on their understanding and students' perception on the learning unit. The effectiveness of the learning unit on Earth geometry and trigonometry is assessed and discussed. Next chapter we also discuss the limitations of the present study and make recommendations for improving it.

5.1 The Learning Unit on Earth Geometry and Trigonometry

Even though trigonometry is rich in applications, only a few have been utilized to facilitate students' learning. Quinlan (2004) began his trigonometry lesson with an activity that required students to measure the height of the classroom without really doing it. Students were challenged to find a way to use a 45-45-90 set square, which was replaced by a 30-60-90 set square in a subsequent activity, and a few other everyday objects to accomplish the task. Cavanagh (2008) let students measure the gradients and the angles of straight lines and asked whether they could find the gradient of a line given its angle of inclination. While both activities led to students' reinvention of tangent and its application, they are triangular in nature. Students would still have trouble transferring to circle trigonometry when the time comes. To the best of our knowledge, the only research study employing a circular application as an integral part in a trigonometry lesson belonged to Thompson (2007) whose starting activity involved modeling the uniform circular motion of a point on the circumference of a Ferris wheel. Students were allowed to freely choose two quantities deemed to be important in describing this circular motion and then had to find the relationship between the two quantities without being informed about the topic of the lesson. At the end of the activity, several groups produced graphs of trigonometric functions but mistook them for graphs of parabolas. Nonetheless, in a traditional trigonometry lesson that followed, the students could easily relate trigonometric constructs to their models. So the *model of* a Ferris wheel became the *model for* various concepts of trigonometry. For them, the study of trigonometry was not disconnected from their lived-in experience, but was closely tied to real-world phenomena.

In this research study, we developed a supplemental instructional unit on trigonometry that centered upon latitude and longitude, a topic belonging to geography and often taught in school in an almost mathematics-free context. Being officially included in curricula confirms the importance and usefulness of the concepts of latitude and longitude in making sense of the world. Unfortunately, latitude and longitude logic requires not just two-, but three-dimensional representations, rendering it unsuitable as the first activity in trigonometry class. Nevertheless, its rich structure and realness should allow students to be immersed in a properly designed activity. As pointed out by the late Ralph P. Boas, Jr. (1912–1992): *there is not time for enough practice on each new topic; and even if there were, it would be insufferably dull … he gets this practice in less distasteful form by studying more advanced mathematics* (Boas, 1957). At the very least, our instruction should provide students with necessary practice based on real-life situations and prepare them for more advanced courses. More importantly, they should be able to enhance their understanding of trigonometry through thinking about and applying it to meaningful real-world problems.

The learning unit on “The Earth and Trigonometry” promoted students’ conceptual understanding of the latitude and longitude and, to a certain extent, of great circle distance with a little help by basic trigonometric and geometric concepts. The evidence for the enhancement not only was in the significant increase in the posttest scores, when compared to the pretest ones, but also in students’ reflection on what they had learned. The success of the unit was due to the combination of the three components, that is, the physical models plus illustrations, the guided-inquiry and cooperative approach and the successive improvement of the pilot studies to yield such an instructional unit: we had to conduct the thought experiment and exploited students’ feedbacks to revise and improve them several times. The research reported here is just the latest version following several pilot studies on students of various schools and backgrounds. After each pilot study we analyzed the data collected and discussed before coming up with modifications of both the physical models and the learning process for the subsequent study.

From our earlier pilot studies before the final one reported here, the following modifications were made in responses to problems arising:

(1) Most students could convert degree into radian; however, they did not know the basis for the relationship. We therefore put in the relationship between angle, as radian, subtending an arc, the radians and the arc length of a circle. Then students found out that the length of the circumference is 2π radian which corresponds to 360^0 along the equator and the meridian. Therefore the relationship between radian and degree in measuring angles was established.

(2) Given the latitude and longitude numbers of a point, students at first could not locate where it should be on the transparent globe model. We therefore asked them to draw many latitude and longitude lines on the transparent surface and pointed out to them that the angles of these lines originated from the center of the globe. Then they grasped the concepts.

(3) Since the transparent sphere (globe) had an equatorial plane with the angles written on it but still without the lines in (2) above, we asked the students to use a half-circumference protractor to measure the latitude and they made the mistake of using it to measure the longitude. However, when a quarter-circumference arc (Figure 3.3(b) from Chapter III, p. 45) protractor (adapted from the half- circumference one) was given fewer students made the mistake.

In addition to the above, the following physical models and illustrations were made to supplement the above changes as follows:

(1) An angle ruler showing correspondence between radian and degree (similar measures to the ruler with cm. and inch) (Figure 3.3(a) from Chapter III, p. 45) was provided so students could see the relationship more clearly.

(2) An illustration showing angles of the latitude and longitude and also showing projection from a point onto the Cartesian coordinates.

Upon the implementation of the latest pilot and actual study, guidance and scaffolding by the teacher was essential in evoking students' previous knowledge and emphasizing the relevance to the real world by using both the physical (visual) models and the illustrative drawings. Students with different background knowledge and visuo-spatial ability had to be brought to a certain level of understanding about the lines on a map, airline flight path, time zone and so forth by a knowledgeable and enthusiastic teacher acting as a facilitator, who communicated constantly with group numbers who needed assistance, albeit only as a guide. After every activity, teachers needed to use the debriefing process to organize the students to review and discuss their experience. Towers and Simmt (2007, p. 252), suggested the implications for teaching that "*debriefings ought to offer rich opportunities to explore students' thinking (i.e., hear the diversity of ideas), but that such thinking must be taken up by the teacher and other students*".

Thus one key factor for the success was in the ability of the teacher to expertly conduct the guide-inquiry learning process; other things being equal, a teacher who knows the content better is more able in handling the whole process (Fernandez, Cannon & Chokshi, 2003; Burghes & Robinson, 2009). In this study, the teacher aroused students' interest with the globe and geography and then led the students simultaneously to an authentic problem in every life about possible flight paths of an airliner to a destination. The students then thought of trigonometry in a more meaningful way (Schiefele & Csikszentmihalyi, 1995). The teacher also facilitated in making students to work cooperatively with their groupmates in all activities of the learning unit. The students were encouraged to conduct their own activities, pose questions, analyze and interpret data, as well as discuss and reach conclusions from their findings (Jarrett, 1997; Gialamas, Cherif, Keller and Hansen, 2000). Each group of students had to utilize the lessons learned from each activity in answering questions in the given worksheet that probed their understanding. That the guided-inquiry approach promotes students' learning, especially in mathematics, has been reported by several researchers (Diezmann, 2004; Brown, Wilson & Fitzallen, 2007). Gialamas, Cherif, Keller and Hansen (2000) suggested that the guided-inquiry approach be used to promote students' understanding of the mathematical concepts through active learning.

It is believed that the positive perception of the learning unit by more than 90% of the students from both schools played an important role in making students learn more mindfully and enthusiastically. This is not surprising that a number of research works have shown that an attractive physical model can arouse students' interest and thus their meaningful learning (Hulett, Williams, Twitty & Turner, 2004; Furner, Yahya & Duffy, 2005). Thompson (1994) reported that concrete materials can be an effective aid to students' thinking and to the instructor's teaching. But effectiveness is contingent upon what the teacher is trying to achieve.

The better understanding of the earth's latitude, longitude, distance between two points on the earth surface, and even the great circle could be partially attributed to the transparent hand-held sphere and hemisphere which are analogous to the globe and its two hemispheres. Gradually, the students were guided to see that the numbers for the latitude arise from the angle subtended by the line drawn from the centre of the globe and the equatorial plane increasing northward or southward; whereas those for the longitude are relative to a north-south line (the prime meridian at 0^0) perpendicular to the equatorial plane and increase eastward or westward. Students were led to see the distance between two points on the earth's curve surface could not be derived accurately by Pythagorean geometry, which deals only with things in a plane. Instead their basic concepts of trigonometry, i. e., sine, cosine, tangent and their respective arc functions can be made use of to obtain the shortest distance between two points on the globe. Concerning the latter, the radius and the angle (in radian) of a circle learned from basic trigonometry and geometry, came into play. Students were asked to make their own spheres of various diameters, and prove to themselves that the relationship between radius, angle in radian and arc length still holds.

To ensure mastering of basics such as the great-circle distance and the angle numbers (east-west) for the longitude and (north-south) for the latitude, students were asked to work on points on the transparent globe model, in addition to worksheet1, to review their conceptual understanding of these angular values. Students also had to compare lengths along various longitudinal and latitudinal arcs for finally calculating the great-circle distance. However, the path along the great circle as a concept for covering the shortest distance between two points on the Earth's surface necessitated more effort at demonstration using various possible examples. Students had to be shown various possibilities and teacher had to answer questions from doubters.

Our results that students had gained their understanding through the use of physical models was reported in the research works of Sowell (1989) and Bayram (2004). They showed that physical models and illustrations are of great help to some whose learning depends more on physical realities than the just mental images. Some students prefer handling concrete and visual objects and then learn better with them.

It should be noticed that the academic background, especially in mathematics, of students from school B was much poorer than that of school A. This was clearly shown in pretest scores in basic trigonometry. It was anticipated at the beginning that students' achievement in both schools might be different. Nevertheless, after participating in the learning unit the previous low achieving students in school B were brought to almost the same level of the medium achievers in school A. However, this was only true for the first three topics on basic trigonometry, latitude and longitude and applied trigonometry, but not "great circle distance" which still eluded them. This was judged from the mean scores from both posttest and worksheet. One explanation might be that the students in school B lacked background knowledge, especially, in basic trigonometry. Although they were brought up to a comparable level in their posttest to those in school A, they might not have sufficient basic mathematical concepts to solve problems of the great circle distance. Another contributing factor might be that students in school B had no experience in three-dimensional perception and thus lacked such visuo-spatial capability required for activity on the great circle distance. This resulted in even poorer understanding of such topics as supported by some works on 3-D (Arcavi, 2003; Kikas, 2006). The conclusions of Kikas' study (2006, p. 280) are "*the different importance of visuo-spatial and verbal abilities in learning before and in school. It may be assumed that the role of verbal abilities becomes even more visible in higher grades when more abstract topics are taught. Scientific theories are abstracted from empirical contexts, and are mediated by signs (verbally or using models, schemas, pictures). People struggle with understanding because they have to integrate visually and verbally perceived fragments of the world".* To ensure that students would understand important concepts of geography above and be able to use them in their higher education, the teacher had to try very hard to explain to them repeatedly after completing all activities, albeit some with limited success.

However, the mean scores of the students' group worksheets in school B were higher than those in school A on the second topic on latitude and longitude. One explanation might be that the manipulatives and illustrations were appropriate in helping to build up the low achieving students' understanding of mathematical concepts (Perry & Howard, 1997).

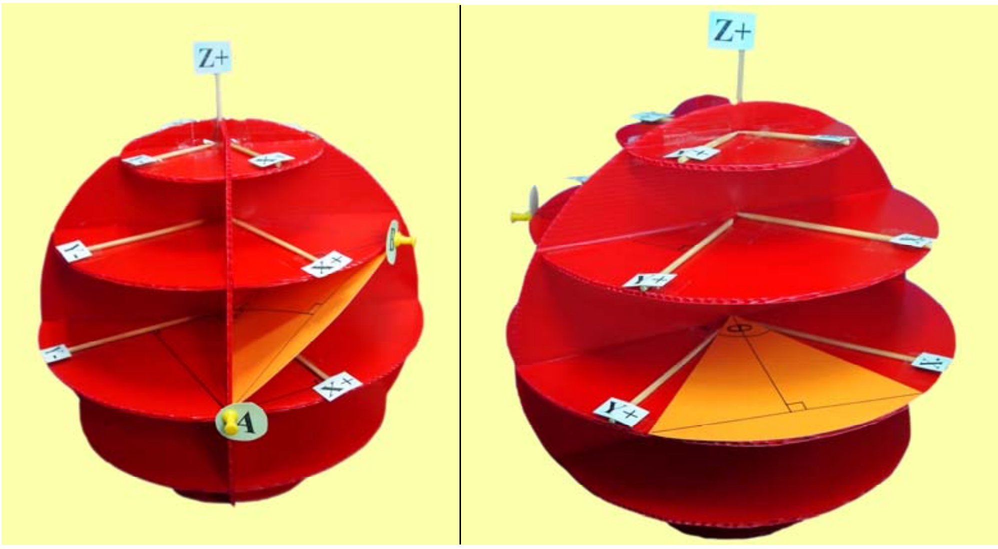
The newly developed learning unit was intended for use as a supplement: it was not designed as a main lesson. It was not our intension to compare this learning unit with the one in the standard curriculum. Thus no control group was employed in this study. Some students had suggested including the learning unit to the standard curriculum. This is because they viewed the learning unit as a new type of instructional approach that would allow them to have a more active role in the learning process. Additionally, they enjoyed working cooperatively as much as using the physical models as a tool for better understanding. This learning unit, however, needs some improvement before being generalized.

5.2 Possible Benefits for Teachers from the Lesson Learned by the Research

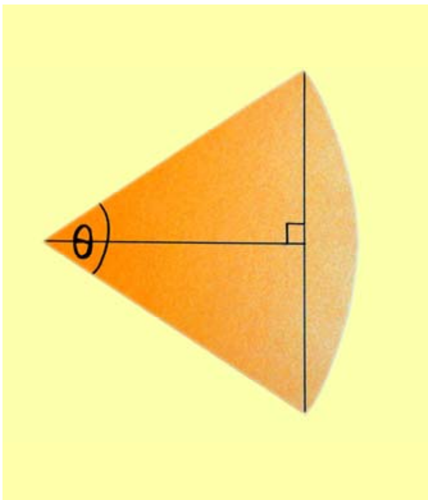
5.2.1 Modification of the Quinlan's model

We now think that the Quinlan's model (2006) slightly modified with x, y, z coordinates together with an xy plane and yz plane will be an extra help (Figure 5.1). This modified physical model together with computer graphics showing animation would be helpful for students to understand the topic of the great-circle distance, especially, for those less willing to imagine. Students knew that the shortest distance between two points on the Earth surface is found in the arc of the segment that passes through them (as in Figure 5.1(a) and Figure 5.1(b)). The teacher then drew the circular sector that arc represents the great-circle path between *A* and *B* (as in Figure 5.1(c)). If the Cartesian coordinates of points *A* and *B* be (x_1, y_1, z_1) and (x_2, y_2, z_2), it was expected that students would see that to obtain the arc length, they had to obtain the straight line or chord (linking the two points, $\overline{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$) subtended by the angle from the center first

($\theta = 2\arcsin\left(\frac{\overline{AB}}{2R_{Earth}}\right)$). To find the arc length, they had to know the Cartesian coordinates in 3-D and substitute the values in a formula for obtaining the arc length. Ability to find the arc length eases the finding of the great-circle distance.



(a)Great-circle distance (b) Comparing great-circle distance to the equator



(c) Circular sector

Figure 5.1 Modification of the Quinlan's model

Animated computer models have been shown by several researchers to facilitate students' understanding because of the movement and the virtual 3-D effects of light intensity contrast and colors (Vincent & McCrae, 2001; Nikolic, 2007;

Durmus & Karakirik, 2006). The lesson plan should also be modified to serve the needs for students with poor background in basic trigonometry. The lesson learned from this research study may be of beneficial to schoolteachers or educators in developing their own instructional model.

5.2.2 A closed-form formula for the great-circle distance

Specialist Mathematics teachers may challenge their students to find a closed-form formula for the great-circle distance between *A* and *B* using trigonometric identities. (The formula is easier to obtain using the scalar product of vectors from the origin to the two points but students should not have learnt about that yet.) If the (latitude, longitude) of *A* and *B* are (φ_A, λ_A) and (φ_B, λ_B), their Cartesian coordinates would be ($\cos\varphi_A \cdot \cos\lambda_A$, $\cos\varphi_A \cdot \sin\lambda_A$, $\sin\varphi_A$) and ($\cos\varphi_B \cdot \cos\lambda_B$, $\cos\varphi_B \cdot \sin\lambda_B$, $\sin\varphi_B$) after normalizing the radius of the Earth to 1. Applying Pythagoras' theorem twice yields

$$\overline{AB} = \sqrt{(\cos\varphi_A \cos\lambda_A - \cos\varphi_B \cos\lambda_B)^2 + (\cos\varphi_A \sin\lambda_A - \cos\varphi_B \sin\lambda_B)^2 + (\sin\varphi_A - \sin\varphi_B)^2}\,.$$

Expanding the first parenthesis yields

$$\cos^2\varphi_A \cos^2\lambda_A + \cos^2\varphi_B \cos^2\lambda_B - 2\cos\varphi_A \cos\lambda_A \cos\varphi_B \cos\lambda_B\,, \qquad (1)$$

and the second parenthesis yields

$$\cos^2\varphi_A \sin^2\lambda_A + \cos^2\varphi_B \sin^2\lambda_B - 2\cos\varphi_A \sin\lambda_A \cos\varphi_B \sin\lambda_B\,. \qquad (2)$$

(1) + (2) yields

$$\cos^2\varphi_A + \cos^2\varphi_B - 2\cos\varphi_A \cos\varphi_B (\cos\lambda_A \cos\lambda_B + \sin\lambda_A \sin\lambda_B)$$

$$= \cos^2\varphi_A + \cos^2\varphi_B - 2\cos\varphi_A \cos\varphi_B \cos(\lambda_A - \lambda_B)\,. \qquad (3)$$

Expanding the third parenthesis yields

$$\sin^2\varphi_A + \sin^2\varphi_B - 2\sin\varphi_A \sin\varphi_B\,. \qquad (4)$$

Putting (3) + (4) inside the radical sign yields

$$\overline{AB} = \sqrt{2\left(1 - [\cos\varphi_A \cos\varphi_B \cos(\lambda_A - \lambda_B) + \sin\varphi_A \sin\varphi_B]\right)}\,.$$

Since *R* in Figure 3.9, the circular sector whose arc represents the great-circle path between *A* and *B***,** has been normalized to 1, we have $\sin\frac{\theta}{2} = \frac{\overline{AB}}{2}$, which implies that

$$\theta = 2\arcsin\frac{\overline{AB}}{2} = 2\arcsin\sqrt{\frac{1 - [\cos\varphi_A \cos\varphi_B \cos(\lambda_A - \lambda_B) + \sin\varphi_A \sin\varphi_B]}{2}}\,.$$

From the half-angle formula for sine (in our case $0 \le \theta \le \pi$),

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} \Rightarrow \theta = 2\arcsin\sqrt{\frac{1-\cos\theta}{2}},$$

we have $\cos\theta = \cos\varphi_A \cos\varphi_B \cos(\lambda_A - \lambda_B) + \sin\varphi_A \sin\varphi_B$. Thus the great-circle distance between *A* and *B* is

$$R\arccos[\cos\varphi_A \cos\varphi_B \cos(\lambda_A - \lambda_B) + \sin\varphi_A \sin\varphi_B].$$

Keep in mind that students may obtain other equivalent formulas using different identities. This activity should help them realize that trigonometric identities can be useful in real-life problems whose solutions are not predetermined, unlike a typical textbook question specifying the final expression into which another expression has to be transformed.

For example,

Given the latitude and longitude or two airports as follows:

- Bangkok (Don Muang) International Airport: 13^0 54' 45"N, 100^0 36' 24"E
- Singapore International Airport: 1^0 21' 0"N, 103^0 59' 39"E

Convert these coordinates to decimal degrees (deg + (min + sec/60)/60) and then to radian (× π/ 180). After the conversion, the coordinates become:

- Bangkok (Don Muang) International Airport: 0.2428, 1.7559
- Singapore International Airport: 0.0236, 1.8150

Using the Earth radius near the equator at approximately 6,378 kilometres, the great-circle distance between Bangkok (Don Muang) International Airport and Singapore International Airport is about 1,446.91 kilometres.

5.2.3 Pythagoras versus the great circle

Teachers should devise a problem to test students' understanding of great circles and to help them realize that the Earth surface covering a relatively small area can be estimated by plane geometry whereas that covering a relatively large area cannot. First, let students locate Adelaide, Canberra, and Mackay on a rectangular map wherein a line of latitude and a line of longitude are always straight and perpendicular to each other, for example, a Mercator projection, and provide them with the cities' latitudes and longitudes. We should round off and/or modify the coordinates slightly so that the three cities form a perfect right triangle on the map. Use the modified coordinates to calculate the distances between Adelaide and Canberra, and between Canberra and Mackay. Supply these distances to students and ask them to find the distance between Adelaide and Mackay. Repeat the activity using Adelaide, Tokyo, and Athens. Let students report their results to the class and discuss the resulting distances. Typically, some students will use Pythagoras' theorem to calculate the distances and the discrepancies will be small for the first group of cities but much larger for the second group. If all students happen to use the same method, the teacher would have to play the devil's advocate by showing the calculation and the result of the other method to stimulate the discussion.

CHAPTER VI
CONCLUSION

6.1 The Effectiveness of the Developed Learning Unit

From our experience, we find that teachers of trigonometry usually do not incorporate several aspects of geography, even as an engagement step, into their instruction, nor do those teaching geography incorporate trigonometry into theirs. We have seen that, in order to understand latitude, longitude, distance, area, and other related topics in geography, students should at least be able to recall certain basic aspects of trigonometry. Likewise, Earth geometry provides an excellent opportunity to integrate real-life experience into trigonometric education. The integrated approach to science education in general, and to mathematics education in particular, helps bring various topics to life and heighten students' attention to an otherwise potentially difficult and uninteresting topic.

A supplementary guided-inquiry learning unit on "The Earth and Trigonometry" addressing problems that concerned mainly with the latitude and longitude and the great-circle distance was developed. It was implemented on high-school students for they better understanding of geography. Due to the three-dimensional nature of the problems, several hand-held models and illustrations were developed to aid students' learning of the globe. The results indicated that the learning unit enhanced students' understanding of latitude and longitude, and great-circle distance, with the help of trigonometry. We came to the realization that students with too little knowledge of the trigonometry would have difficulty grasping the concepts of the great-circle distance. Students' reflections indicated the importance of utilizing real-world problems and group activities in the instruction. The positive results in terms of students' improved conceptual understanding and diminished difficulty with the problems above suggested that students had learned better from concrete models and interaction with peers.

6.2 Implication of this Study

The inquiry-based learning unit on “The Earth and Trigonometry” was used to frame the learning unit for integrated mathematics and geography. According to the results in this study, the newly developed learning unit promoted students’ conceptual understanding by applying basic trigonometry and geometry to solve real world problem. Teachers should adopt an inquiry approach, physical models and illustrations to various mathematics topics. The guideline for teachers is to provide real-world problems to encourage students’ achievement and their conceptual understanding of mathematical concepts through activities. Based on the successful of the implementation of the learning unit on Earth geometry and trigonometry, teachers should modify or construct the physical models and illustrations to help students’ visualization for students with other levels. The lesson learned from this study could be used as guideline for school-teachers for effective learning teaching. For example, collaboration and debriefing is an important process in activities for all students. Students should have an opportunity to discuss their findings, answers, and ideas with peers by collaboration in small-group work in their class. At the end of each activity, teacher should explore students’ thinking and to ensure that they have learned through the given activity.

6.3 Limitations of this Study

Although the findings in this study suggested that the inquiry-based learning unit promoted high-school students to achieve the concept of Earth geometry and trigonometry this research has many drawbacks. Even though there was no overall significant difference in conceptual understanding test scores in the two schools, we could discern the differences in the enthusiasm as well as interest and willingness to learn in the lower achieving students. The students from the two schools were different in their prior knowledge in trigonometry. This caused problems in conducting the learning activity with students having poor prior knowledge, especially, in third activity on the great-circle distance. Another drawback is that the number of students and schools were rather small to be generalized. This study should be confirmed with larger numbers of students having different backgrounds. Moreover, physical models and illustrations may limit some students' willingness to imagine without aid.

6.4 Recommendations

The recommendations of this study were divided into two parts. The first part focuses on the recommendations for further study that provided the developed implementations based on the evidences from students' worksheets, students' reflection on their understanding and students' perception on the learning unit. Furthermore, the recommendations came from the researcher and the teachers in school who participated in this learning unit. The others part deals with the recommendations for the further development that prepared the alternative approach to develop the learning unit based on the realistic mathematics education (RME).

6.4.1 Recommendations for further study

Results from students' reflection suggested that some of them confused the given angles located in the second quadrant of the unit circle. Thus we agreed with Kendal and Stacy (1997) that basic trigonometric functions should be introduced using the unit circle in more than one quadrant. The teachers should be trained to facilitate students in their groups, especially on group discussion, a necessary process to support students' thinking during the activities. Moreover, they should be trained to review the background concepts because the benefits of the new mathematical learning unit concern with the integrated concepts between mathematics and geography.

6.4.2 Recommendations for further development

The learning unit could also be modified to be more appropriate for students with poor background on basic trigonometry by adding more steps in the learning activity. Since most difficult topic was the 3-D projection on the three orthogonal axes; x, y, z. Teachers should be trained to slightly modify the Quinlan's model (2006) with x, y, z coordinates together with a xy plane and yz plane in order to be an extra help for the students. At first, teachers should add some exercises for students to convert the latitude and longitude points into the Cartesian coordinates in two dimensions before those in three dimensions. Thus the students should start with converting the points on the equator plane and latitude plane into the values for x and y. Then they should convert the points on any longitude plane into the values for x and z. These added exercises would be leading them from 2-D projection to 3-D projection in the worksheet. Then they would realize that the same amount of difference in degrees along different latitude planes may lead to different values; whereas the same amount of difference in degrees along different longitude plane always leads the same values. Some students suggested that this learning unit should be also added to the Thai mathematics curriculum for high-school students. Nevertheless, the unit needs improvement as suggested above before it can be generalized.

Another suggested improvement concerns the teaching of trigonometry using RME (Realistic Mathematics Education) right at the beginning. In RME, students construct mathematical knowledge by *mathematizing matter from reality* through the use of a model. The model starts off as a *model of* a real-life situation and progressively transforms into a *model for* similar mathematical problems. RME have six principles, namely, *activity, reality, level, intertwinement, interaction, and guidance* principles. In summary, the principles state that students learn mathematics by solving real-life (reality) problems (activity) under the teacher's guidance. In addition, they should have opportunities to share their strategy and creativity with their peers (interaction).

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APPENDICES

APPENDIX A

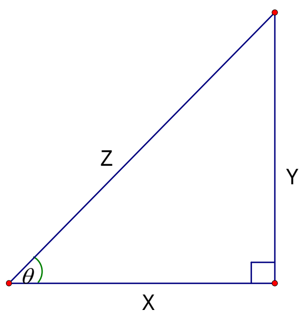
Pretest Questions

on

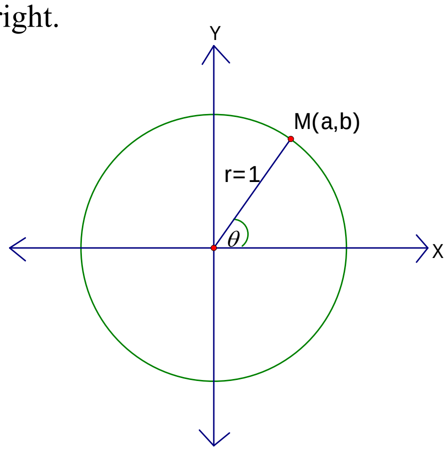
"The Earth and Trigonometry"

Instruction: Fill in the blank. Give reasons

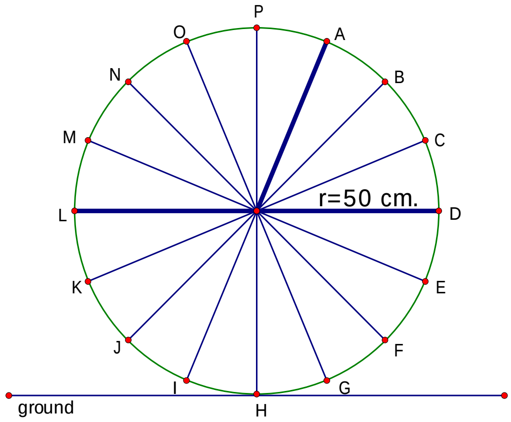
1. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ of the triangle on the right hand side.



2. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ from the figure on the right.



3. A bicycle wheel with spokes A to P



3.1 Based on the wheel, find the shortest distance from A to the spoke $\overline{LD}$ (hint: drop a vertical line from A onto $\overline{LD}$).

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3.2 Based on the wheel, if you only have a string 100 cm. long, how can you find a one radian angle, without aid from any other tools?

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4. The Earth is assumed to be perfectly spherical

4.1 Regarding identical latitudinal displacements along the curved lines of different longitudes, is the latitudinal displacement from 11^0 north to 23^0 north on longitude 50^0 east different from that from 47^0 north to 59^0 north on longitude 170^0 east? Explain

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4.2 Regarding longitudinal displacements (degree displacements) along the curved lines of different latitudes, is the longitudinal displacement from 15^0 east to 23^0 east on latitude 45^0 north different from that from 68^0 east to 76^0 east on latitude 45^0 south? Explain

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5. From the figure on the right,

5.1 Find the coordinates for point A.

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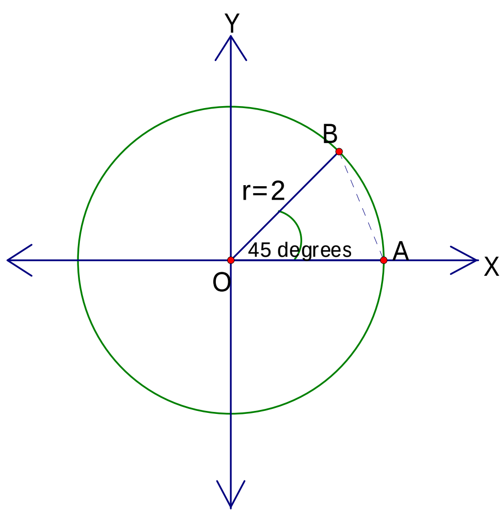

5.2 Find the coordinates for point B.

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5.3 Find the length for chord AB.

5.4 Show clearly how to arrive at the length of the arc AB.

6. Let A and B be points on a spherical globe of radius R with coordinates 30^0 north 45^0 east for A and those for B are 60^0 north 60^0 east.

6.1 Find the Cartesian coordinates for points A and B.

Let the origin be at the center of the globe and x, y, z are mutually perpendicular axes. Along the *z*-axis, values are positive toward the North Pole. Along the *x*-axis, the values for both latitude and longitude are 0^0. The y-axis is perpendicular to the xz plane at the latitude 0^0 and longitude of 90^0 east.

6.2 Find the length of the straight line AB.

6.3 Show the way to find the arc length which is the shortest distance for AB on the surface of the globe.

APPENDIX B

Posttest Questions

on

"The Earth and Trigonometry"

Instruction: Fill in the blank. Give reasons

1. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ of the triangle on the right hand side.

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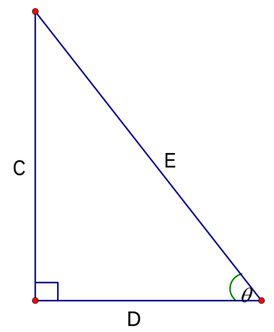

2. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ from the figure on the right.

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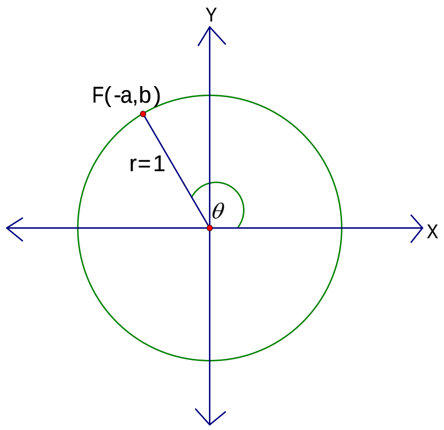
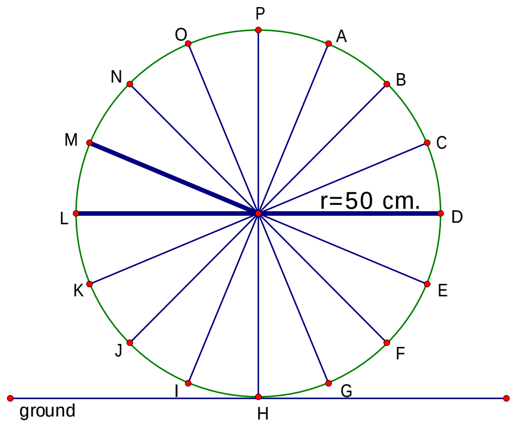


3. A bicycle wheel with spokes A to P



3.1 Based on the wheel, find the shortest distance from M to the spoke $\overline{LD}$ (hint: drop a vertical line from M onto $\overline{LD}$).

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3.2 Based on the wheel, if you only have a string 100 cm. long, how can you find two radian angles, without aid from any other tools?

4. The Earth is assumed to be perfectly spherical

4.1 Regarding identical latitudinal displacements along the curved lines of different longitudes, is the latitudinal displacement from 2^0 north to 12^0 north on longitude 10^0 east different from that from 67^0 north to 77^0 north on longitude 150^0 east? Explain

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4.2 Regarding longitudinal displacements (degree displacements) along the curved lines of different latitudes, is the longitudinal displacement from 5^0 east to 25^0 east on latitude 30^0 north different from that from 136^0 east to 156^0 east on latitude 60^0 north? Explain

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5. From the figure on the right, let $\alpha = 45^0$ and $\beta = 120^0$

5.1 Find the coordinates for point A.

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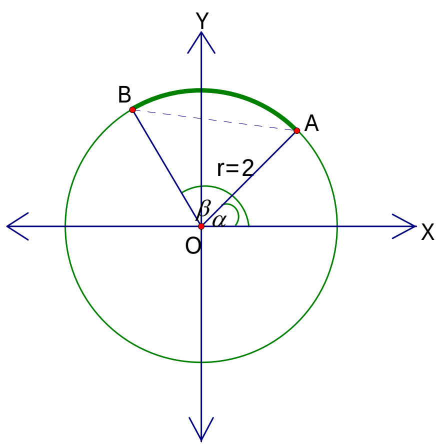

5.2 Find the coordinates for point B.

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5.3 Find the length for chord AB.

5.4 Show clearly how to arrive at the length of the arc AB.

6. Let A and B be points on a spherical globe of radius R with coordinates 30^0 north 30^0 east for A and those for B are 45^0 north 60^0 east.

6.1 Find the Cartesian coordinates for points A and B.

Let the origin be at the center of the globe and x, y, z are mutually perpendicular axes. Along the *z*-axis, values are positive toward the North Pole. Along the *x*-axis, the values for both latitude and longitude are 0^0. The y-axis is perpendicular to the xz plane at the latitude 0^0 and longitude of 90^0 east.

6.2 Find the length of the straight line AB.

6.3 Show the way to find the arc length which is the shortest distance for AB on the surface of the globe.

APPENDIX C

Answers to Pretest

on

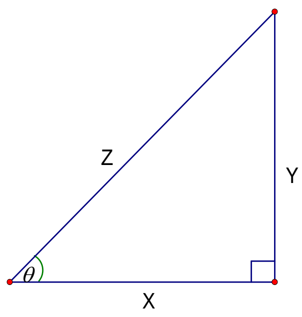
"The Earth and Trigonometry"

1. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ of the triangle on the right hand side.

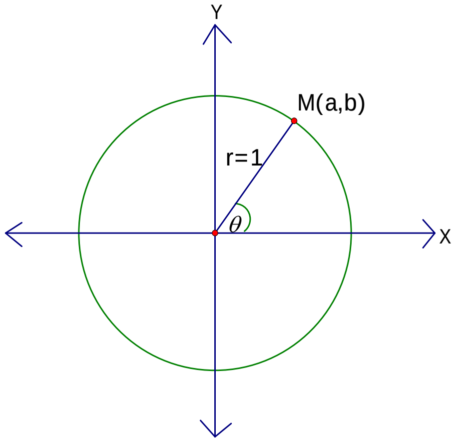
Solution $\sin\theta = \frac{Y}{Z}$

$\cos\theta = \frac{X}{Z}$

$\tan\theta = \frac{Y}{X}$



2. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ from the figure on the right.

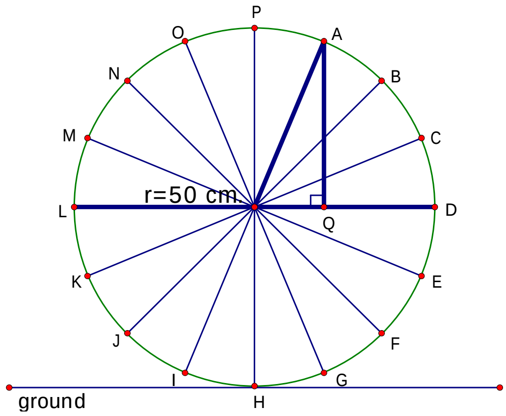


Solution $\sin\theta = b$

$\cos\theta = a$

$\tan\theta = \frac{b}{a}$

3. A bicycle wheel with spokes A to P



3.1 Based on the wheel, find the shortest distance from A to the spoke $\overline{LD}$ (hint: drop a vertical line from A onto $\overline{LD}$).

Suggested solution 1

Drop a vertical line from A to point Q on $\overline{LD}$. The angle between adjacent spokes is 22.5^{o}. Since there are three pairs of spokes between A and D, the angle is $3\times 22.5^{o} = 67.5^{o}$.

$$\text{Thus } \sin 67.5^{0} = \frac{\overline{AQ}}{50} \Rightarrow \boxed{\overline{AQ} = 50\sin 67.5^{0}}$$

Suggested solution 2

Because the angle formed by spokes A and D is 67.5^{o}. Therefore the angle formed at point A should be $180^{0} - \left(90^{0} + 67.5^{0}\right) = 22.5^{0}$.

$$\text{Thus } \cos 22.5^{0} = \frac{\overline{AQ}}{50} \Rightarrow \boxed{\overline{AQ} = 50\cos 22.5^{0}}$$

3.2 Based on the wheel, if you only have a string 100 cm. long, how can you find a one radian angle, without aid from any other tools?

Solution

Find a string equaling the length of the spoke which measures at 50 cm. Place the string around the edge of the wheel. The two radii linking the two ends of the string should make an angle of one radian.

4. The Earth is assumed to be perfectly spherical

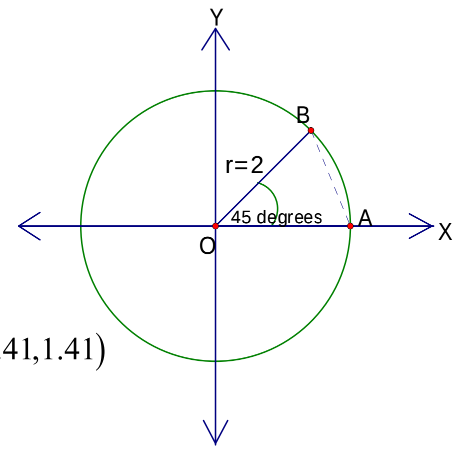
4.1 Regarding identical latitudinal displacements along the curved lines of different longitudes, is the latitudinal displacement from 11^0 north to 23^0 north on longitude 50^0 east different from that from 47^0 north to 59^0 north on longitude 170^0 east? Explain

Solution The two arc lengths should be equal because along any longitude the displacement is proportional to the angle value.

4.2 Regarding longitudinal displacements (degree displacements) along the curved lines of different latitudes, is the longitudinal displacement from 15^0 east to 23^0 east on latitude 45^0 north different from that from 68^0 east to 76^0 east on latitude 45^0 south? Explain

Solution Not different, because the two latitude lines have the same absolute values in degree. So the two circles have the same circumference and in this case arc length for the same degree change in longitude.

5. From the figure on the right,



5.1 Find the coordinates for point A.

Solution

$$\left(2\cos 0^{\circ}, 2\sin 0^{\circ}\right) = (2,0)$$

5.2 Find the coordinates for point B.

Solution

$$\left(2\cos 45^{\circ}, 2\sin 45^{\circ}\right) = \left(\sqrt{2}, \sqrt{2}\right) = (1.41, 1.41)$$

5.3 Find the length for chord AB.

Suggested solution 1

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{\left(2-\sqrt{2}\right)^2 + \left(0-\sqrt{2}\right)^2}$$

$$= \sqrt{\left(4-4\sqrt{2}+2\right)+(2)}$$

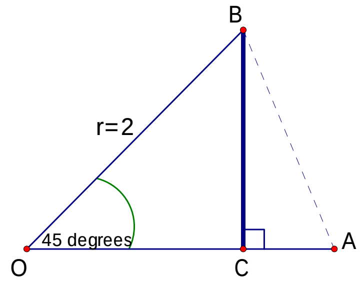
$$= \boxed{\sqrt{8-4\sqrt{2}}}$$

Or $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(2-1.41)^2 + (0-1.41)^2} = \sqrt{0.35+1.99} = \boxed{1.53}$

Suggested solution 2

$$\left(\overline{AB}\right)^2 = \left(\overline{OA}\right)^2 + \left(\overline{OB}\right)^2 - 2\left(\overline{OA}\right)\left(\overline{OB}\right)\cos\theta$$
$$= 2^2 + 2^2 - 2(2)(2)\cos 45^o$$
$$= 8 - 8\left(\frac{\sqrt{2}}{2}\right)$$
$$= 8 - 4\sqrt{2}$$

Suggested solution 3 Focus on the inscribed triangle, drop a vertical line $\overline{BC}$ onto $\overline{OA}$ at point C.



Since $O\hat{A}B = O\hat{B}A = \frac{180^0 - 45^0}{2} = 67.5^0$

Therefore $\sin O\hat{A}B = \frac{\overline{BC}}{\overline{AB}} \Rightarrow \sin 67.5^0 = \frac{\sqrt{2}}{\overline{AB}} \Rightarrow \overline{AB} = \frac{\sqrt{2}}{\sin 67.5^0}$

5.4 Show clearly how to arrive at the length of the arc AB.

Suggested solution 1

The arc length for the 180^0 angle is $\pi\mathbf{r}=\mathbf{2}\pi$.

Therefore the arc length for the 45^0 angle is $\frac{2\pi}{180^o} \times 45^o \approx \boxed{1.57}$.

Suggested solution 2

The circumference is $2\pi r = 2 \times \frac{22}{7} \times 2 = \frac{88}{7}$.

A 45^0 angle divide the circumference into 8 equal parts.

Therefore the arc length AB is $\frac{88}{7 \times 8} = \frac{11}{7}$.

6. Let A and B be points on a spherical globe of radius R with coordinates 30^0 north 45^0 east for A and those for B are 60^0 north 60^0 east.

6.1 Find the Cartesian coordinates for points A and B.

Let the origin be at the center of the globe and x, y, z are mutually perpendicular axes. Along the *z*-axis, values are positive toward the North Pole. Along the *x*-axis, the values for both latitude and longitude are 0^0. The y-axis is perpendicular to the xz plane at the latitude 0^0 and longitude of 90^0 east.

Hints

$$\begin{aligned} A &= \left(R\cos\frac{\pi}{6}\cos\frac{\pi}{4}, R\cos\frac{\pi}{6}\sin\frac{\pi}{4}, R\sin\frac{\pi}{6} \right) \\ &= \left(\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}R, \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}R, \frac{1}{2}R \right) = (0.61R, 0.61R, 0.50R) \end{aligned}$$

$$\begin{aligned} B &= \left(R\cos\frac{\pi}{3}\cos\frac{\pi}{3}, R\cos\frac{\pi}{3}\sin\frac{\pi}{3}, R\sin\frac{\pi}{3} \right) \\ &= \left(\frac{1}{2}\times\frac{1}{2}R, \frac{1}{2}\times\frac{\sqrt{3}}{2}R, \frac{\sqrt{3}}{2}R \right) = (0.25R, 0.43R, 0.87R) \end{aligned}$$

6.2 Find the length of the straight line AB.

Hints

$$\begin{aligned} &\sqrt{(0.61R-0.25R)^2+(0.61R-0.43R)^2+(0.50R-0.87R)^2} \\ &= \sqrt{0.1296R^2+0.0324R^2+0.1369R^2} \\ &= 0.5467R \end{aligned}$$

6.3 Show the way to find the arc length which is the shortest distance for AB on the surface of the globe.

Hints

Let θ be the angle subtending the arc AB.

Since $\sin\frac{\theta}{2} = \frac{\overline{AB}}{2R} \Rightarrow \frac{\theta}{2} = \arcsin\frac{\overline{AB}}{2R} \Rightarrow \theta = 2\arcsin\frac{\overline{AB}}{2R}$

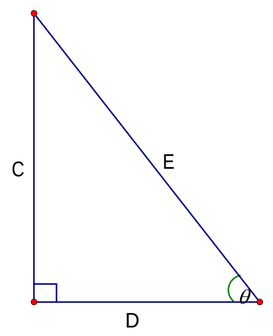
Leading to $\theta = 2\arcsin\frac{0.5467R}{2R} = 0.55$

Therefore the shortest arc passing through the two points is 0.55R unit length.

APPENDIX D

Answers to Posttest
on
"The Earth and Trigonometry"

1. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ of the triangle on the right hand side.

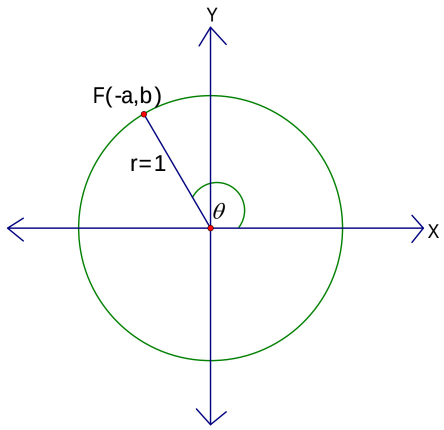


Solution $\sin\theta = \frac{C}{E}$

$\cos\theta = \frac{D}{E}$

$\tan\theta = \frac{C}{D}$

2. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ from the figure on the right.

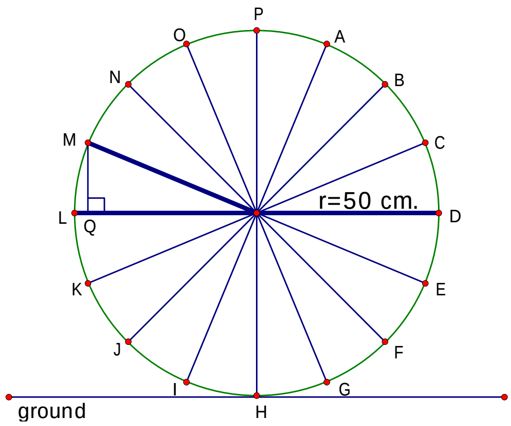


Solution $\sin\theta = b$

$\cos\theta = -a$

$\tan\theta = -\frac{b}{a}$

3. A bicycle wheel with spokes A to P



3.1 Based on the wheel, find the shortest distance from M to the spoke $\overline{LD}$ (hint: drop a vertical line from M onto $\overline{LD}$).

Solution

Drop a vertical line from M to point Q on $\overline{LD}$. The angle between adjacent spokes is 22.5° .

$$\text{Thus } \sin 22.5^{0} = \frac{\overline{MQ}}{50} \Rightarrow \boxed{\overline{MQ} = 50\sin 22.5^{0}}$$

3.2 Based on the wheel, if you only have a string 100 cm. long, how can you find two radian angles, without aid from any other tools?

Solution

Find a string equaling the length of the spoke which measures at 50 cm. Place the string around the edge of the wheel. The two radii linking the two ends of the string should make an angle of two radians.

4. The Earth is assumed to be perfectly spherical

4.1 Regarding identical latitudinal displacements along the curved lines of different longitudes, is the latitudinal displacement from 2^{0} north to 12^{0} north on longitude 10^{0} east different from that from 67^{0} north to 77^{0} north on longitude 150^{0} east? Explain

Solution

The two arc lengths should be equal because along any longitude the displacement is proportional to the angle value.

4.2 Regarding longitudinal displacements (degree displacements) along the curved lines of different latitudes, is the longitudinal displacement from 5^0 east to 25^0 east on latitude 30^0 north different from that from 136^0 east to 156^0 east on latitude 60^0 north? Explain

Solution

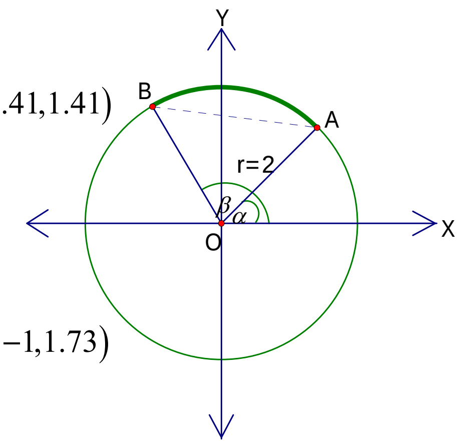
The one along 30^0 latitudinal line is longer because the displacement between longitudinal lines become shorter as they go toward the poles.

5. From the figure on the right, let $\alpha = 45^0$ and $\beta = 120^0$

5.3 Find the coordinates for point A.

Solution

$$\left(2\cos 45^{o}, 2\sin 45^{o}\right) = \left(\sqrt{2}, \sqrt{2}\right) = (1.41, 1.41)$$



5.4 Find the coordinates for point B.

Solution

$$\left(2\cos 120^{o}, 2\sin 120^{o}\right) = \left(-1, \sqrt{3}\right) = (-1, 1.73)$$

5.3 Find the length for chord AB.

Suggested solution 1

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{\left(\sqrt{2}+1\right)^2 + \left(\sqrt{2}-\sqrt{3}\right)^2}$$
$$= \sqrt{\left(3+2\sqrt{2}\right) + \left(5-2\sqrt{2}\sqrt{3}\right)}$$
$$= \boxed{\sqrt{8+2\sqrt{2}\left(1-\sqrt{3}\right)}}$$

Or

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{(1.41+1)^2 + (1.41-1.73)^2}$$
$$= \sqrt{5.81 + 0.1024}$$
$$\approx \boxed{2.54}$$

Suggested solution 2

$$\left(\overline{AB}\right)^2 = \left(\overline{OA}\right)^2 + \left(\overline{OB}\right)^2 - 2\left(\overline{OA}\right)\left(\overline{OB}\right)\cos\theta$$
$$= 2^2 + 2^2 - 2(2)(2)\cos\left(120^\circ - 45^\circ\right)$$
$$= 8 - 8\cos\left(75^\circ\right)$$
$$= 8 - 8(0.2588)$$
$$= 5.9296$$
$$\boxed{\overline{AB} \approx 2.46}$$

5.4 Show clearly how to arrive at the length of the arc AB.

Solution

The arc length for the 180^0 angle is $\boldsymbol{\pi r = 2\pi}$.

Therefore the arc length for the 75^0 angle is $\frac{2\pi}{180^\circ} \times 75^\circ \approx \boxed{2.62}$.

6. Let A and B be points on a spherical globe of radius R with coordinates 30^0 north 30^0 east for A and those for B are 45^0 north 60^0 east.

6.1 Find the Cartesian coordinates for points A and B.

Let the origin be at the center of the globe and x, y, z are mutually perpendicular axes. Along the *z*-axis, values are positive toward the North Pole. Along the *x*-axis, the values for both latitude and longitude are 0^0. The y-axis is perpendicular to the xz plane at the latitude 0^0 and longitude of 90^0 east.

Hints

$$A = \left(R\cos\frac{\pi}{6}\cos\frac{\pi}{6}, R\cos\frac{\pi}{6}\sin\frac{\pi}{6}, R\sin\frac{\pi}{6}\right)$$
$$= \left(\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}R, \frac{\sqrt{3}}{2}\times\frac{1}{2}R, \frac{1}{2}R\right)$$
$$= (0.75R, 0.43R, 0.50R)$$

$$B = \left(R\cos\frac{\pi}{4}\cos\frac{\pi}{3}, R\cos\frac{\pi}{4}\sin\frac{\pi}{3}, R\sin\frac{\pi}{4}\right)$$
$$= \left(\frac{\sqrt{2}}{2}\times\frac{1}{2}R, \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2}R, \frac{\sqrt{2}}{2}R\right)$$
$$= (0.35R, 0.61R, 0.71R)$$

6.2 Find the length of the straight line AB.

Hints

$$\sqrt{(0.75R-0.35R)^2+(0.43R-0.61R)^2+(0.50R-0.71R)^2}$$
$$=\sqrt{0.16R^2+0.0324R^2+0.0441R^2}$$
$$=0.4863R$$

6.3 Show the way to find the arc length which is the shortest distance for AB on the surface of the globe.

Hints

Let θ be the angle subtending the arc AB.

Since $\sin\frac{\theta}{2}=\frac{\overline{AB}}{2R}\Rightarrow\frac{\theta}{2}=\arcsin\frac{\overline{AB}}{2R}\Rightarrow\theta=2\arcsin\frac{\overline{AB}}{2R}$

Leading to $\theta=2\arcsin\frac{0.4863R}{2R}=0.49$

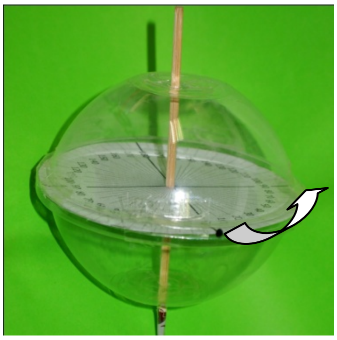
Therefore the shortest arc passing through the two points is 0.49R unit length.

APPENDIX E

Worksheet 1 with Solutions
on
"Basic Trigonometry and Latitude and Longitude"

Instruction: You should work with your friends in explaining the following:-

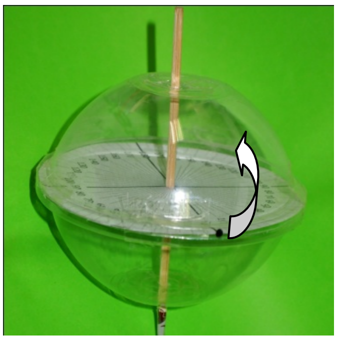
1. Hold the Earth model (with the transparent top and showing angle values) and turn the marked point toward you. Measure a distance (to your right) of 2 radians along the Equator and make a mark with a magic marker.



2. What is the angle value (in degree) from 1 above? (Use the Earth model)

Answer about 115^0

3. Measure a distance of 1 radian from the given mark toward where the stick protrudes from the model and make it with marker.



4. What would the angle value be from 3? Use the quarter-circumference piece.

Answer about 57.5^0

5. What the relationship between the angle values and the lengths in 1 and 3?

Answer The angle values are proportional to the radian values.

6. If change the size of the Earth model, do you think the radian degree ruler or quarter-circumference piece provided are still usable. Why?

Answer The size of the ruler and quarter-circumference piece should vary with the size of the Earth model.

7. From 6 above, the altered Earth model with the corresponding altered ruler and quarter-circumference piece, should you expect to find changes in the angle values both in radian and degree?

Answer With the corresponding adjustments, there should be no change.

8. How would you make a radian ruler to measure angles on an Earth atlas?

Construction

Step 1 Find the Earth's radius which is equivalent to 1 radian arc length.

Step 2 Find the angle values in degrees

π is by definition the ratio of the circumference to the diameter (twice the radius).

Symbolically written as $\pi = \frac{C}{2r} \Rightarrow C = (2\pi)r$.

The students should be able to conclude that the circumference is 2π radians long. And the arc length subtended by 360^0 angle is 2π radians long.

The arc length subtended by an angle θ is $\left(\frac{2\pi}{360^0}\right) \times \theta^0 = \left(\frac{\pi\theta}{180}\right)$ radians.

The angle of one radian is then defined by an angle subtending an arc length of one radius.

Therefore $\left(\frac{\pi\theta}{180}\right)$ radians subtend an arc length of $\left(\frac{\pi\theta}{180}\right) \times r$.

Given r of 3.7 cm., the arc length

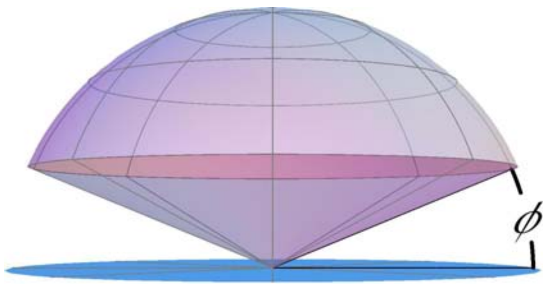
For $30^0 = \frac{2\pi}{360^0} \times \theta^0 \times \text{r} = \frac{\pi}{180} \times 30 \times 3.7 \approx 1.9\,\text{cm.}$

For $45^0 = \frac{2\pi}{360^0} \times \theta^0 \times \text{r} = \frac{\pi}{180} \times 45 \times 3.7 \approx 2.9\,\text{cm.}$

For $60^0 = \frac{2\pi}{360^0} \times \theta^0 \times \text{r} = \frac{\pi}{180} \times 60 \times 3.7 \approx 3.9\,\text{cm.}$

9. Draw lines for latitude 30^0 and 60^0 north on the transparent plastic dome of the Earth model.

10. How does one measure the latitude angle?



Answer The angle of a latitude line is measured from the equatorial plane. It is the top circumference of a symmetrical funnel. Any point on this circumference has this angle be it north or south reaching a maximum of 90^0.

11. The latitude lines, are they of the same length, why?

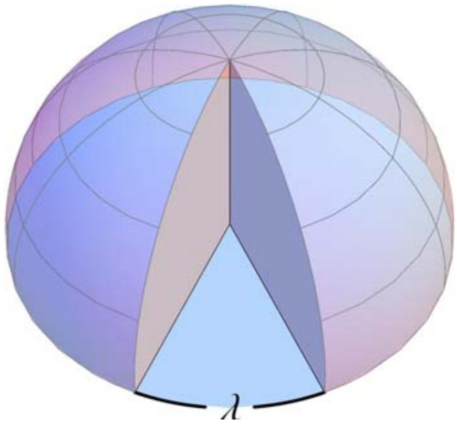
Answer Different latitude lines have different angle values and/or signs. Latitude lines with the same absolute values are of the same length. Otherwise they are not.

12. What are latitude planes?

Answer They are circular planes parallel to the equatorial planes and differ in size depending on the angle.

13. Draw longitude lines of 30^0 and 120^0 east on the Earth model.

14. How do you measure the longitude angle?



Answer A longitude of a point is the angle measured from the half circle of the prime meridian to the half circle of the meridian passing through that point.

15. What are longitude lines, are they all equally long?

Answer Longitudinal lines run along the great circles joining the North and South Poles, they are all of the same length.

16. What are the longitudinal planes, what are they like?

Answer Longitudinal planes are circular and of the same diameter.

17. Assuming a perfectly spherical Earth, put points (as instructed) on the Earth surface and compare and contrast the arc lengths between two pairs of points.

17.1 Compare two longitudinal lines between two parallel latitude lines, one from point 45^0 north, 60^0 east to another from 45^0 south, 60^0 east and another from point 45^0 north, 30^0 east to point 45^0 south, 30^0 east

Answer The arc lengths are equal because the latitudinal displacements are identical for the same amount of longitudinal degree difference.

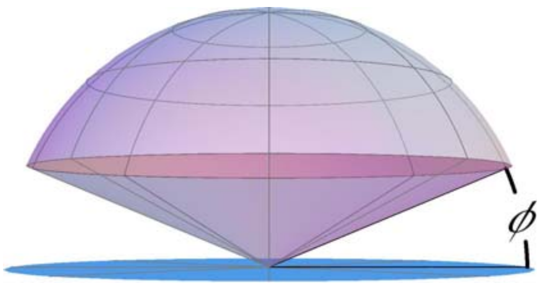
17.2 Compare two latitudinal lines between two longitudinal lines, one from 60^0 north, 30^0 east to 60^0 north, 60^0 east and another from 30^0 north, 30^0 east to 30^0 north, 60^0 east

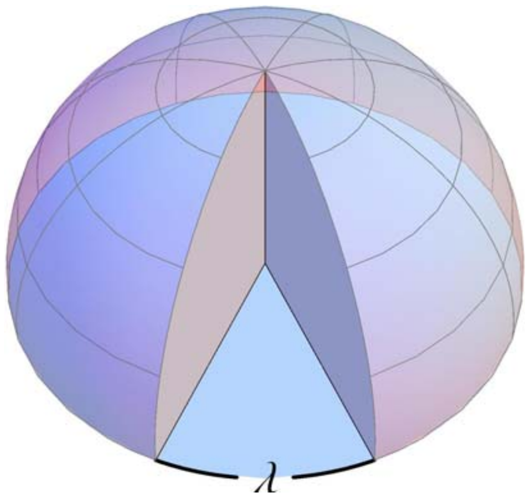
Answer The one along 30^0 latitudinal line is longer because the displacement between longitudinal lines become shorter as they go toward the poles.

APPENDIX F

Worksheet 2 with Solutions
on
"Applied Trigonometry"

Instruction: You should work with your friends in explaining (drawing is allowed) the following:-





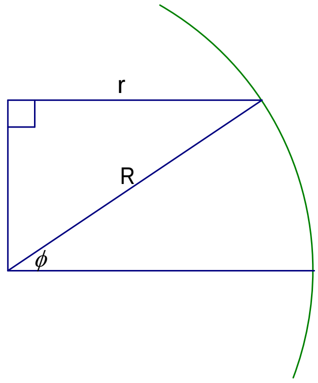
1. Show calculation for the arc length between points on a longitudinal line.

Answer Let ϕ be the radian angle change of the latitudinal points, and the longitudinal radius (Earth's radius) is R.

Therefore the arc length between the two points longitudinally is $R\phi$.

2. Show calculation for the arc length between points on a latitudinal line.

Answer



Let r be the radius of latitudinal plane with an angle value of ϕ. It can be seen that $\cos\phi = \frac{r}{R} \Rightarrow \boxed{r = R\cos\phi}$.

Let λ is the radian change of the longitudinal lines.

Therefore the latitudinal arc length at ϕ angle is $r\lambda$.

3. Let A be at 60^0 north and 30^0 degree east and B be at 60^0 north and 150^0 degree west. Find the distances between A and B along the latitudinal line and longitudinal line passing both points through the North Pole to see whether they are different? If they are different, why?

Answer Convert A and B positions into radians first:-

Point A is at $\frac{\pi}{3}$ radian north, $\frac{\pi}{6}$ radian east and point B is at $\frac{\pi}{3}$ radian north, $\frac{5\pi}{6}$ radian west.

Hints Let R be the Earth's radius and ϕ be the latitudinal change in radian.

Then the arc length along the longitude equals $R\phi$.

Find ϕ: A and B are on the same latitudinal plane, but on different sides of the Prime Meridian.

The latitudinal angle of A is $\frac{\pi}{3}$ and the arc length from the equatorial plane is $\frac{\pi}{3}R$.

The latitudinal angle of B is $\frac{\pi}{3}$ and the arc length from the equatorial plane is $\frac{\pi}{3}\text{R}$.

Therefore, the arc length along the longitude line joining the two points passing the North Pole is $\pi\text{R} - \left(\frac{\pi\text{R}}{3} + \frac{\pi\text{R}}{3}\right) = \frac{\pi\text{R}}{3}$.

Hints Let R be the Earth's radius,

r is the radius of the latitudinal plane at $\frac{\pi}{3}$, and

λ is the radian angle change along the longitude.

Here $\text{r} = \text{R}\cos\frac{\pi}{3} = \frac{\text{R}}{2}$ and $\lambda = \frac{\pi}{6} + \frac{5\pi}{6} = \pi$.

The arc length along the latitude is $\frac{\text{R}}{2} \times \pi = \frac{\pi\text{R}}{2}$.

Therefore, the longitudinal path from A to B is shorter than the latitudinal path by $\frac{\pi\text{R}}{2} - \frac{\pi\text{R}}{3} = \frac{\pi\text{R}}{6}$.

4. Can you find a distance between A and B on the Earth surface shorter than in 3?

Answer No (You have to look a pie in the next session).

APPENDIX G

Worksheet 3 with Solutions
on
"Great-circle Distance"

Instruction: Work in groups to explain and draw pictures to answer the following questions.

1. Without the longitudinal and latitudinal lines on the Earth, can you exploit the pie illustrated to help you make one for the model with the shortest arc length?

Answer Draw two straight lines from the Earth center to the two points, the pie is the plane contained by the spherical envelope.

2. For two points on the Earth surface (not necessarily on the same longitudes and latitudes), can you make a pie with the shortest arc joining them.

Answer Same as in 1.

3. The spherical Earth has a radius of R units. If A is at 0^0 north, 0^0 east and B is at 60^0 north, 150^0 west based on the center of the Earth, express A and B in Cartesian coordinates.

   The z axis is positive toward the north and negative toward the south.

   The positive x axis goes through 0^0 longitude and 0^0 latitude.

   The y axis is positive toward 0^0 latitude and 90^0 east longitude.

Solution

$$A = (\cos 0 \cos 0, \cos 0 \sin 0, \sin 0) = (1\times 1, 1\times 0, 0) = (1,0,0)$$

$$B = \left(\cos\frac{\pi}{3}\cos\frac{5\pi}{6}, \cos\frac{\pi}{3}\sin\frac{5\pi}{6}, \sin\frac{\pi}{3}\right) = \left(\frac{1}{2}\times\left(-\frac{\sqrt{3}}{2}\right), \frac{1}{2}\times\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = (-0.43, 0.25, 0.87)$$

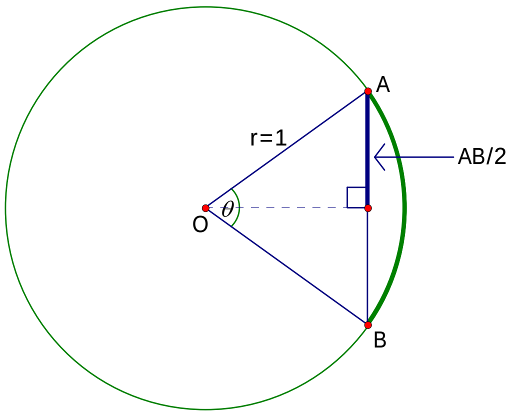
4. From 3 show how you find the distance between A and B.

Solution The Cartesian straight line between A and B has the length of

$$
\begin{aligned}
AB &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \\
&= \sqrt{(1 + 0.43)^2 + (0 - 0.25)^2 + (0 - 0.87)^2} \\
&= \sqrt{2.0449 + 0.0625 + 0.7569} \\
&= 1.6924
\end{aligned}
$$

5. From 3 show how you find a pie with the shortest arc passing through the two points.

Solution Let θ be the angle subtending the arc AB.



Then

$$
\sin\frac{\theta}{2} = \frac{AB}{2}
$$

$$
\Rightarrow \frac{\theta}{2} = \arcsin\frac{AB}{2}
$$

$$
\Rightarrow \theta = 2\arcsin\frac{AB}{2} \quad \_\_\_\_\_\_\_\_\_\_ (*)
$$

Substituting the length of the straight line between A and B in equation (*) to find the angle subtending the arc AB, we obtain

$$
\theta = 2\arcsin\frac{AB}{2} = 2\arcsin\left(\frac{1.6924}{2}\right) = \boxed{2.0176}
$$

6. From 5 show how you work out the arc length of the pie.

Solution One radian is an angle subtending an arc of R unit.

Therefore 2.0176 radians subtends the arc length of 2.0176R

The arc length of the pie is then 2.0176R.

BIOGRAPHY

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