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IMPROVEMENT OF FINITE ELEMENT MESH QUALITY BY USING GEOMETRICAL QUALITY MEASURES AND OPTIMIZATION

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This paper discusses possible procedures to improve finite element meshes in order to enable an accurate and stable numerical analysis processes. In general, the process of mesh improvement has to address three main aspects: mesh untangling (removal or fix of inverted finite elements), improvement of element shape, and making the element sizes more uniform. This paper focusses on the last two aspects: improving shape and size uniformity. This problem is addressed from purely geometrical aspect and by engaging optimization methods; thus, stress/strain related finite element quality measures are not considered. Two various element quality measures are discussed with emphasis on their coupling with an adequate optimization procedure. These two measures are the *inverse mean ratio* measure and the *size* measure. The first one addresses the shape and size of finite elements, but concentrates more on the shape. The later one, on the other hand, addresses only finite element size. Since mesh improvement tasks are typical multi-objective optimization problems, the paper also addresses briefly the procedure how to transform the multi-objective problem into a usual single-objective one that can be solved by employing standard optimization techniques. In this work a gradient-based approximation method was employed to do the optimization. The discussed theory is numerically tested on simply deformed meshes.

Keywords: IMR measure, Size measure, Numerical modeling.

1 INTRODUCTION

Many engineering and scientific applications require a discretization of the geometric domain by using a finite element mesh. At this point it is important to note that the mesh quality has a huge impact on the solution process and the obtained results. In extreme cases mesh distortion can result even in a failure of the numerical solution process, caused, for example, by an ill-conditioned system matrix or by simple divergence of a solution process. On the other side, minor mesh distortion will not be reflected in the solution process, but may seriously distort the computed result. Anyhow, mesh quality absolutely needs to be observed and corrected/improved if necessary. A good mesh should be absolutely without inverted elements, the element proportions should be within reasonable limits, and the element sizes should be rather uniform. To ensure such properties of the mesh, various methods and quality measures have already been proposed Jibum *et al*. (2013), Escobar *et al*. (2003), Munson (2007), Shewchuk (2002) and Novak (2013).

In this article, two of such quality measures are discussed: the inverse mean ratio measure and the size measure. While the first one primarily addresses the shape and size of finite elements (but concentrates more on the shape), the second one addresses only finite element size. By using a suitable transformation, the mesh improvement task is formulated as a single-objective optimization problem that can be solved by employing standard optimization techniques. For its solution, a gradient-based approximation method was employed.

2 QUALITY MEASURES OF THE MESH

Two methods are discussed:

1) IMR is the inverse mean ratio measure [3]

2) Size measure [4]

2.1 IMR

IMR is defined by Eq. (1):

$$q_{IMR} = \frac{\sqrt{\mathrm{tr}((\mathbf{CW}^{-1})^{\mathrm{T}}\ (\mathbf{CW}^{-1}))}}{2\left|\det(\mathbf{CW}^{-1})\right|} \tag{1}$$

where **W** is a matrix corresponding to the ideal-shape element and **C** measures the difference between ideal and deformed shape of the element. The ideal shape for a 3-nodes element is an equilateral triangle. The matrix **W** is composed from the vectors pointing from node 1 to 2 and from node 1 to 3 of the triangle. In Eq. (2), (a, b, and c) are the nodes of the triangular element:

$$\mathbf{W} = \begin{bmatrix} 1 & 0{,}5 \\ 0 & 0{,}86 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} b-a & c-a \end{bmatrix} \tag{2}$$

The value of the IMR is between one and infinite. The value higher than one means, that the ideal and tested element have different shapes.

2.2 Size Measure

The objective of this metric is to address the size of finite elements. This metric is defined as:

$$q_{SIZE} = \frac{A}{A_{\lim}} \tag{3}$$

where A is the area of the actual element and $A_{\min}$ is the reference area. The values of this metric are between 0 and infinity.

3 MULTI-OBJECTIVE OPTIMIZATION

The optimization procedures involving more than one objective function are commonly addressed as a multi-objective optimization. Such optimization problems do not have a single solution unique minimum. Therefore, the Pareto set is a commonly used concept for determining the optimal solutions of such a task.

Let define **F** to be a vector of n objective functions:

$$\mathbf{F}(x) = [F_1\ (x), F_2\ (x), \ldots F_n\ (x)]^{\mathrm{T}} \tag{4}$$

The point $x^* \in X$ is a Pareto optimum, if there is no point $x \in X$, such that $F(x) \le F(x^*)$ and $F_i(x) \le F_i(x^*)$ for at least one function. This means that there is no other point that would improve at least one objective function without worsening the other ones. Typically, there are many points satisfying the Pareto condition; these points define the Pareto set.

Searching for the whole Pareto set is a very difficult undertaking. Therefore, a commonly used approach is to transform the multi-objective optimization problem into a usual problem involving only one objective function. For this purpose, the following approaches are used in this paper:

1) Exponential sum:

$$F = c\ \ln[\sum_{i=1}^{n} \mathrm{e}^{F_i/c}] \tag{5}$$

where F_i is the i-th objective function and $c > 0$ is a control parameter. Typical values for c are between 10^4 and 10^6.

2) Equivalent sum of objective function squares:

$$F = \sum_{i=1}^{n} F_i^2 \tag{6}$$

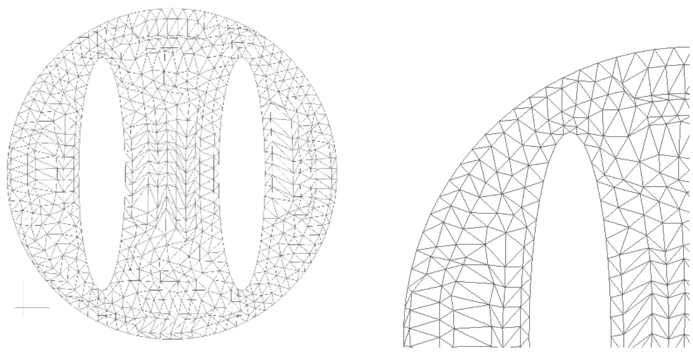
4 NUMERICAL RESULTS OF OPTIMIZATION METHODS TESTING

The analysis of a 3-node finite element mesh was done by employing the IMR and the size measure, as well as both measures joined into the multi-objective function. The multi-objective function of the exponential sums is formulated as:

$$F = c\ \ln(e^{\frac{1}{c}(|q_{SIZE}-1|-(q_{SIZE}-1))} + e^{\frac{1}{c}q_{IMR}}) \tag{7}$$

where c is a regulation parameter [4]. In the second case the multi-objective function of the equivalent sum of squares is formulated as:

$$F = q_{SIZE}^2 + q_{IMR}^2 \tag{8}$$



(a) (b)

Figure 1. Initial distorted mesh of 2D disc (a); Part of the disc (b).

By employing the Abaqus application, both, the shape and the size of more than half of the elements were distorted in such a way that all elements remained valid (non-inverted). The considered geometrical domain contained 1,336 triangle elements (Figure 1).

Figures 2 and 3 show the histories of the optimization procedure for average quality of the element, according to shape measure (IMR), and average quality of the element according to size measure.

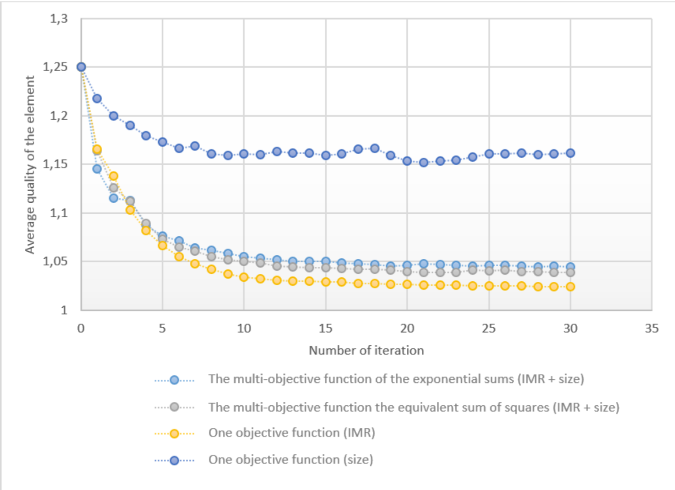


Figure 2. Average quality of the element according to shape measure (IMR).

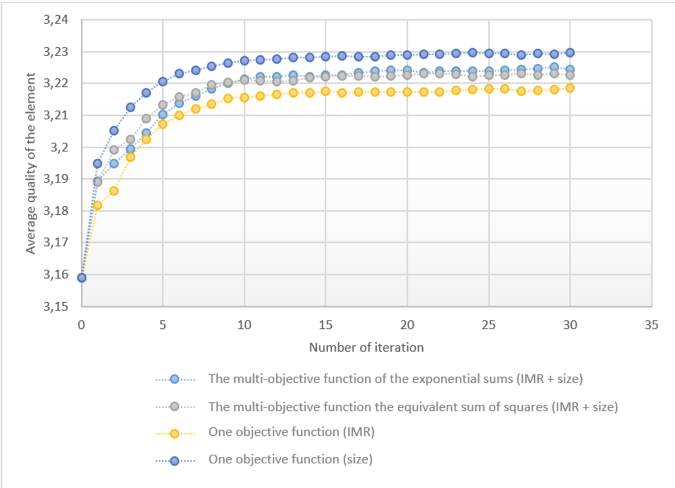


Figure 3. Average quality of the element according to size measure.

It is evident that the single-objective size function produces the poorest results (in relation to good average element shape) of all the procedures when the quality of the element is measured by shape (IMR) metrics (Figure 2). Accordingly, the single-objective IMR function produces the worst results (in relation to uniform element size) of all the procedures when the quality of the element is measured by size metrics (Figure 3). On the other side, the multi-objective approach proved to be efficient in both cases – since it produced a mesh with good element shapes and uniform sizes. The ideal element for IMR metrics is an equilateral triangle;

therefore, we expected that the final mesh will have this kind of elements when the single-objective IMR method is used. When the single-objective size method is used on the other hand, the elements of different shapes and same area dimension might be formed. A relevant conclusion would be that the mesh optimization towards the uniform size has the negative effect on the element shape.

Figures 4 and 5 are presenting the quality of the poorest element, measured by the IMR or size metrics. It is not unusual that a single element of very poor quality causes bad results in a finite elements analysis.

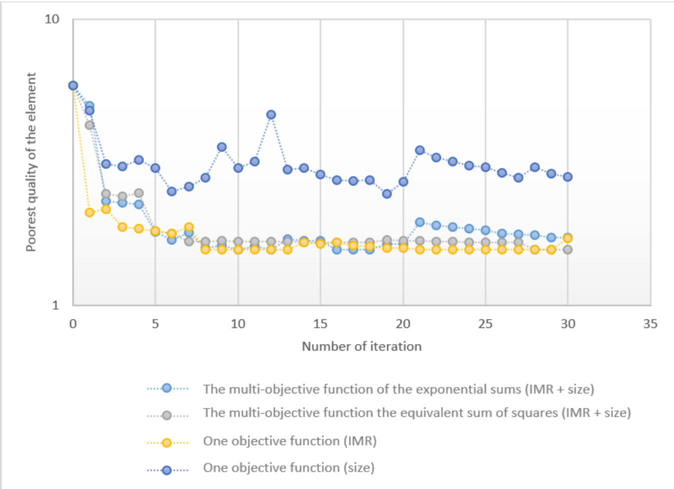


Figure 4. Poorest quality of the element measured by the shape (IMR).

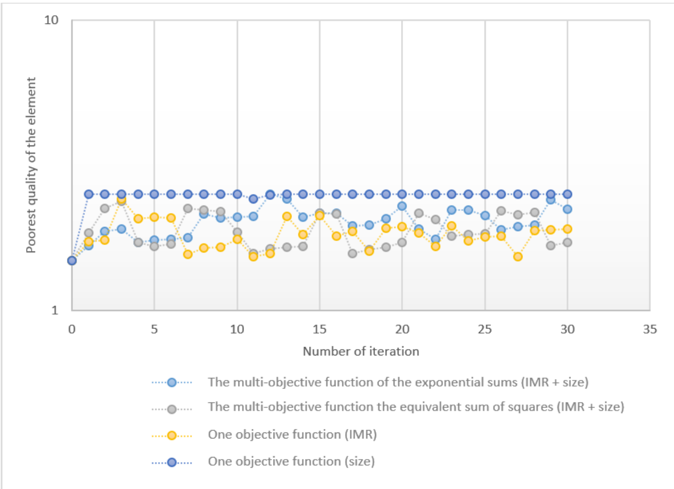
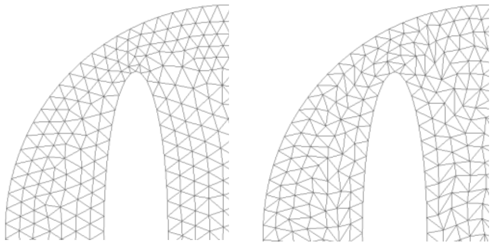


Figure 5. Poorest quality of the element measured by the size.

Figure 4 demonstrates the weakness of the single-objective size method. Generated elements with the same area dimension are implicitly producing elements of poor shape quality, as confirmed by the IMR metrics. The effective elimination of the poor shape elements takes place when employing the single-objective IMR method as well the multi-objective approach. Figure 5 shows that single-objective size method is quick and effective in improving the non-uniformed elements sizes.

Higher value of size metrics means better quality. As expected, the single-objective IMR method is not performing well in generating the elements of the uniform size, since it is focused on the shape. The multi-objective method of equivalent sums is efficient in adapting the elements of non-uniform size. The evident variability of the multi-objective function is a consequence of the simultaneous improving of both mesh quality measures.

As expected, the single-objective shape IMR method performs best in ensuring good element shapes. Figures 6 and 7 are showing the meshes produced by different kind of optimizations.



(a) (b)

Figure 6. Single-objective method: (a) with IMR metrics; (b) with size metrics.

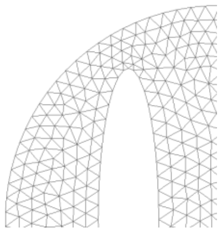


Figure 7. Multi-objective method of equivalent sums.

Observing the multi-objective optimization and referring to the presented cases, we can conclude that coupling the objective functions makes a good sense in some situations. The multi-objective optimization approach has improved the mesh in the sense of the best compromise between shape and size.

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